Lecture 35: Calculus with Parametric Curves

Let $\mathcal{C}$ be a parametric curve described by the parametric equations $x = f(t), y = g(t)$. If the function f and g are differentiable and y is also a differentiable function of x, the three derivatives $\frac{dy}{dx}$, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are related by the Chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

using this we can obtain the formula to compute $\frac{dy}{dx}$ from $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \qquad \frac{dx}{dt} \neq 0}$$

1. The value of $\frac{dy}{dx}$ gives gives the slope of a tangent to the curve at any given point. This sometimes helps us to draw the graph of the curve.
2. The curve has a **horizontal tangent** when $\frac{dy}{dx} = 0$, and has a **vertical tangent** when $\frac{dy}{dx} = \infty$.

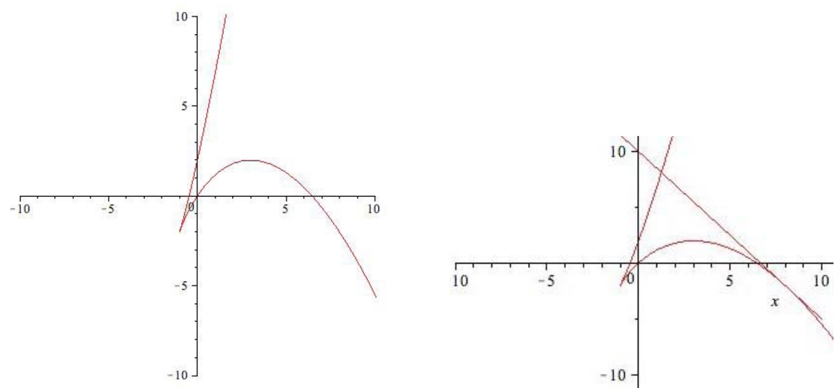
The second derivative $\frac{d^2y}{dx^2}$ can be obtained as well from $\frac{dy}{dx}$ and $\frac{dx}{dt}$. Indeed,

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \text{if} \qquad \frac{dx}{dt} \neq 0}$$

Notice the lack of symmetry, to find $\frac{d^2y}{dx^2}$ we divide by the derivative of $\frac{dx}{dt}$ and we do not use the derivative $\frac{dy}{dt}$.

Example 1 (a) Find an equation of the tangent to the curve

$$x = t^2 - 2t \qquad y = t^3 - 3t \qquad \text{when} \qquad t = -2$$



(b) Find the points on the curve where the tangent is horizontal

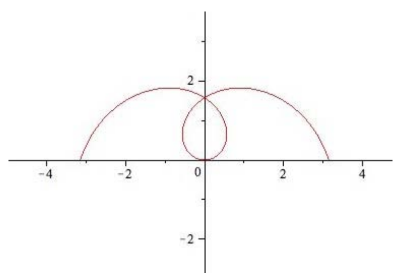
(c) Does the curve have a vertical tangent?

(d) Use the second derivative to determine where the graph is concave up and concave down.

Example 2 Consider the curve $\mathcal{C}$ defined by the parametric equations

$$x = t\cos t \qquad y = t\sin t \qquad -\pi \leq t \leq \pi$$

Find the equations of both tangents to $\mathcal{C}$ at $(0, \frac{\pi}{2})$



Area under a curve

Recall that the area under the curve $y = F(x)$ where $a \leq x \leq b$ and $F(x) > 0$ is given by

$$\int_a^b F(x)dx$$

If this curve can be traced by parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$ then we can calculate the area under the curve by computing the integral:

$$\boxed{\int_\alpha^\beta g(t)f'(t)dt \qquad \text{or} \qquad \int_\beta^\alpha g(t)f'(t)dt}$$

Example Find the area under the curve

$$x = 2\cos t \qquad y = 3\sin t \qquad 0 \leq t \leq \frac{\pi}{2}$$

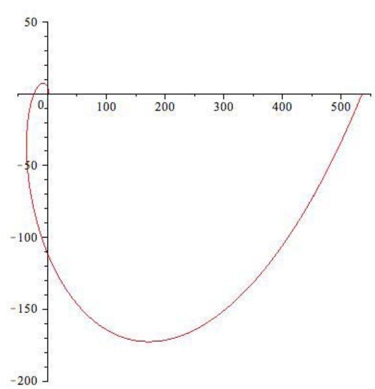
Arc Length: Length of a curve

If a curve $\mathcal{C}$ is given by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where the derivatives of f and g are continuous in the interval $\alpha \leq t \leq \beta$ and $\mathcal{C}$ is traversed exactly once as t increases from α to β, then we can compute the length of the curve with the following integral:

$$\boxed{L = \int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt}$$

Example Find the arc length of the spiral defined by

$$x = e^t\cos t \qquad y = e^t\sin t \qquad 0 \leq t \leq 2\pi$$



Example Find the arc length of the circle defined by

$$x = \cos 2t \qquad y = \sin 2t \qquad 0 \le t \le 2\pi$$

Do you see any problems?