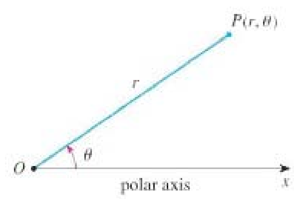
Lecture 36: Polar Coordinates

A polar coordinate system, gives the co-ordinates of a point with reference to a point O and a half line or ray starting at the point O. We will look at polar coordinates for points in the xy-plane, using the origin $(0, 0)$ and the positive x-axis for reference.



A point P in the plane, has polar coordinates (r, θ), where r is the distance of the point from the origin and θ is the angle that the ray $|OP|$ makes with the positive x-axis.

Example 1 Plot the points whose polar coordinates are given by

$$(2, \frac{\pi}{4}) \quad (3, -\frac{\pi}{4}) \quad (3, \frac{7\pi}{4}) \quad (2, \frac{5\pi}{2})$$

Note the representation of a point in polar coordinates is not unique. For instance for any θ the point $(0, \theta)$ represents the pole O. We extend the meaning of polar coordinate to the case when r is negative by agreeing that the two points (r, θ) and $(-r, \theta)$ are in the same line through O and at the same distance $|r|$ but on opposite side of O. Thus

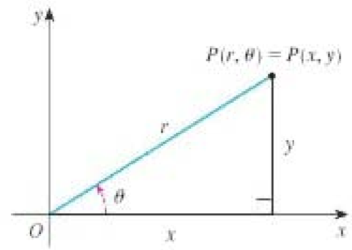
$$(-r, \theta) = (r, \theta + \pi)$$

Example 2 Plot the point $(-3, \frac{3\pi}{4})$

Polar to Cartesian coordinates

To convert from Polar to Cartesian coordinates, we use the identities:

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta}$$



Example 3 Convert the following to Cartesian coordinates $(2, \frac{\pi}{4})$ and $(3, -\frac{\pi}{3})$

Cartesian to Polar coordinates

To convert from Cartesian to polar coordinates, we use the following identities

$$\boxed{r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}}$$

When choosing the value of θ, we must be careful to consider which quadrant the point is in, since for any given number a, there are two angles with $\tan\theta = a$, in the interval $0 \leq \theta \leq 2\pi$.
Example 3 Give polar coordinates for the points $(2, 2)$, $(1, -\sqrt{3})$, and $(-1, \sqrt{3})$

Graphing Equations in Polar Coordinates

The graph of an equation in polar coordinates $r = f(\theta)$ or $F(r, \theta) = 0$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

1. **Lines:** A line through the origin $(0, 0)$ has equation $\theta = \theta_0$
2. **Circle centered at the origin:** A circle of radius r_0 centered at the origin has equation $r = r_0$ in polar coordinates.

Example 4 Graph the following equations $r = 5$, $\theta = \frac{\pi}{4}$

Example 5 Graph the equation $r = 6 \sin \theta$ and convert the equation to an equation in Cartesian coordinates.

Example 6 Sketch the **cardioid** $r = 1 + \cos \theta$

Using Symmetry We have 3 common symmetries in curves which often shorten our work in graphing a curve of the form $r = f(\theta)$:

1. If $f(-\theta) = f(\theta)$ the curve is symmetric about the horizontal line $\theta = 0$. In this case it is enough to draw either the upper or lower half of the curve, drawing the other half by reflecting in the line $\theta = 0$(the x-axis). [Example $r = 1 + \cos\theta$].

2. If $f(\theta) = f(\theta + \pi)$, the curve has central symmetry about the pole or origin. Here it is enough to draw the graph in either the upper or right half plane and then rotate by 180 degree to get the other half. [Example $r = \sin 2\theta$.]

3. If $f(\theta) = f(\pi - \theta)$, the curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$. It is enough to draw either the right half or the left half of the curve in this case. [Example $r = 6\sin\theta$.]

Example 7 Sketch the **rose** $r = \cos(4\theta)$

Circles

We have many equations of circles with polar coordinates: $r = a$ is the circle centered at the origin of radius a, $r = 2a\sin\theta$ is the circle of radius a centered at $(a, \frac{\pi}{2})$ (on the y-axis), and $r = 2a\cos\theta$ is the circle of radius a centered at $(a, 0)$ (on the x-axis).

Tangents to Polar Curves

If we want to find the equation of a tangent line to a curve of the form $r = f(\theta)$, we write the equation of the curve in parametric form, using the parameter θ.

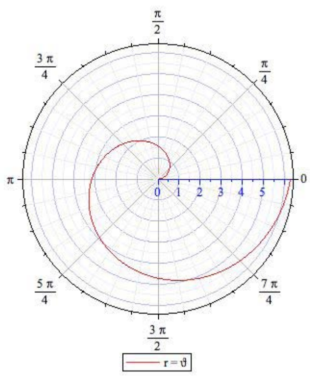
$$\boxed{x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta}$$

From the calculus of parametric equations, we know that if f is differentiable and continuous we have the formula:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Note As usual, we locate horizontal tangents by identifying the points where $dy/dx = 0$ and we locate vertical tangents by identifying the points where $dy/dx = \infty$

Example 8 Find the equation of the tangent to the curve $r = \theta$ when $\theta = \frac{\pi}{2}$



Example 9 (a) Find the equation of the tangent to the circle $r = 6\sin\theta$ when $\theta = \frac{\pi}{3}$

(b) At which points do we have a horizontal tangent?

(c) At which points do we have a vertical tangent?