Lecture 37: Areas and Lengths in Polar Coordinates

Given a polar curve $r = f(\theta)$, we can use the relationship between Cartesian coordinates and Polar coordinates to write **parametric equations** which describe the curve using the parameter θ

$$\boxed{x(\theta) = f(\theta)\cos\theta \qquad y(\theta) = f(\theta)\sin\theta}$$

To compute the arc length of such a curve between $\theta = a$ and $\theta = b$, we need to compute the integral

$$\boxed{L = \int_a^b \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}d\theta}$$

We can simplify this formula because

$$(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 = [f'(\theta)]^2 + [f(\theta)]^2 = r^2 + (\frac{dr}{d\theta})^2$$

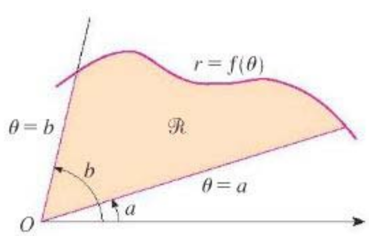
Thus

$$\boxed{L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta}$$

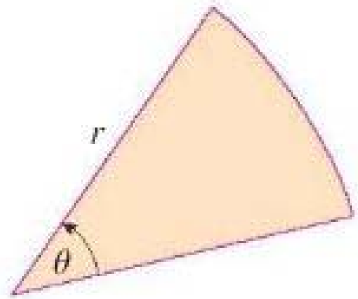
Example 1 Compute the length of the polar curve $r = 6\sin\theta$ for $0 \le \theta \le \pi$

Area in polar coordinates

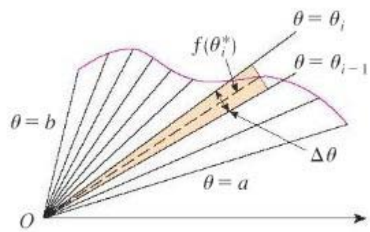
Suppose we are given a polar curve $r = f(\theta)$ and wish to calculate the area swept out by this polar curve between two given angles $\theta = a$ and $\theta = b$. This is the region $\mathcal{R}$ in the picture below:



As always, we divide this shape into smaller pieces, the area of each of which we can calculate. Usually our pieces are rectangles, however, rectangles are not easy to describe in polar coordinates nor are they well adapted the pie-like shape above. Instead, we use small sectors of a circle:



The area of a sector of a circle of radius r with central angle θ is easily seen to be $A = \frac{r^2\theta}{2}$. We now divide the interval $[a, b]$ into n small subintervals of equal width $\Delta\theta$ and in each interval we pick an angle $\theta_1^*, \theta_2^*, \ldots, \theta_n^*$. We can then approximate the shape $\mathcal{R}$ by n small sectors; for each i, we have a sector of a circle of radius $f(\theta_i^*)$ with central angle $\Delta\theta = \frac{b-a}{n}$.



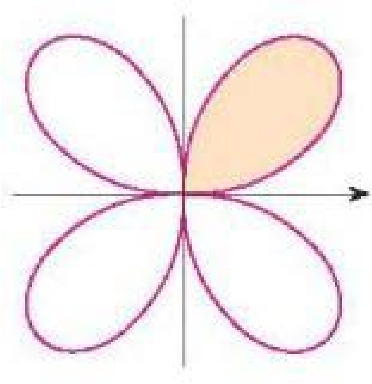
We could then approximate the area of $\mathcal{R}$ by the Riemann sum

$$A \approx \sum_{i=1}^{n} \frac{f(\theta_i^*)^2}{2} \Delta\theta$$

Taking $n \to \infty$ yields the following integral expression of the area

$$\boxed{\int_a^b \frac{f(\theta)^2}{2} d\theta = \int_a^b \frac{r^2}{2} d\theta}$$

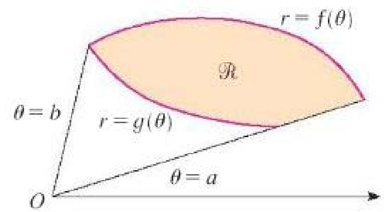
Example 2 Compute the area bounded by the curve $r = \sin 2\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$.



$r = \sin 2\theta$

If instead we consider a region bounded between two polar curves $r = f(\theta)$ and $r = g(\theta)$ then the equations becomes

$$\boxed{\frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta}$$



Example 3 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

NOTE The fact that a single point has many representation in polar coordinates makes it very difficult to find all the points of intersections of two polar curves. It is important to draw the two curves!!!
Example 4 Find all possible intersection points of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.