Lecture 4 : General Logarithms and Exponentials.

For $a > 0$ and x any real number, we define

$$\boxed{a^x = e^{x \ln a}, \quad a > 0.}$$

The function a^x is called the exponential function with base a.
Note that $\ln(a^x) = x \ln a$ is true for all real numbers x and all $a > 0$. (We saw this before for x a rational number).
Note: We have no definition for a^x when $a < 0$, when x is irrational.

For example $2^{\sqrt{2}} = e^{\sqrt{2}\ln 2}$, $2^{-\sqrt{2}}$, $(-2)^{\sqrt{2}}$(no definition).

Algebraic rules

The following **Laws of Exponent** follow from the laws of exponents for the natural exponential function.

$$\boxed{a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x}$$

Proof $a^{x+y} = e^{(x+y)\ln a} = e^{x \ln a + y \ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y$. etc...

Example Simplify $\frac{(a^x)^2 a^{x^2+1}}{a^2}$.

Differentiation

The following **differentiation rules** also follow from the rules of differentiation for the natural exponential.

$$\boxed{\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = a^x \ln a \qquad \frac{d}{dx}(a^{g(x)}) = \frac{d}{dx} e^{g(x)\ln a} = g'(x) a^{g(x)} \ln a}$$

Example Differentiate the following function:

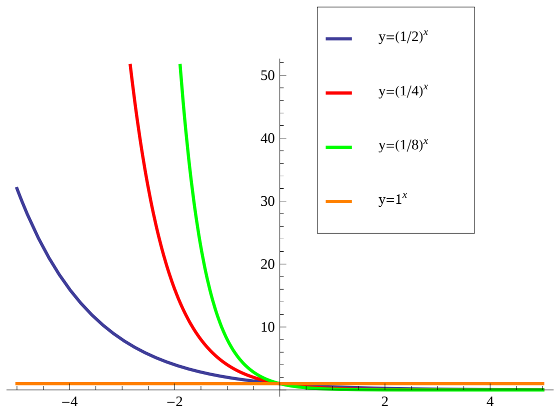
$$f(x) = (1000)2^{x^2+1}.$$

Graphs of Exponential functions. Case 1: $0 < a < 1$

- y-intercept: The y-intercept is given by $y = a^0 = e^{0 \ln a} = e^0 = 1$.
- x-intercept: The values of $a^x = e^{x \ln a}$ are always positive and there is no x intercept.

- Slope: If $0 < a < 1$, the graph of $y = a^x$ has a negative slope and is always decreasing, $\frac{d}{dx}(a^x) = a^x \ln a < 0$. In this case a smaller value of a gives a steeper curve.
- The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x(\ln a)^2 > 0$.
- As $x \to \infty$, $x \ln a$ approaches $-\infty$, since $\ln a < 0$ and therefore $a^x = e^{x \ln a} \to 0$.
- As $x \to -\infty$, $x \ln a$ approaches ∞, since both x and $\ln a$ are less than 0. Therefore $a^x = e^{x \ln a} \to \infty$.

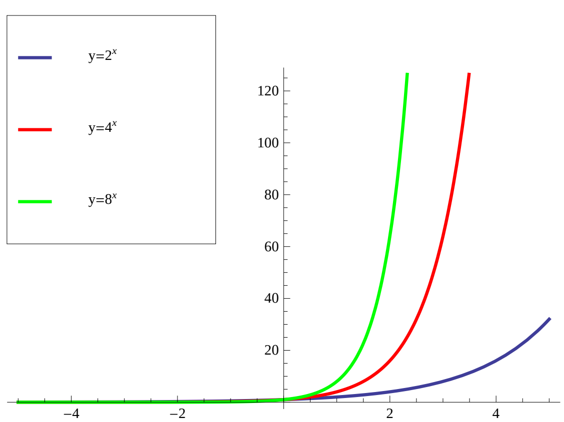
$$\boxed{\text{For } \; 0 < a < 1, \quad \lim_{x \to \infty} a^x = 0, \quad \lim_{x \to -\infty} a^x = \infty}.$$



Graphs of Exponential functions. Case 2: $a > 1$

- y-intercept: The y-intercept is given by $y = a^0 = e^{0 \ln a} = e^0 = 1$.
- x-intercept: The values of $a^x = e^{x \ln a}$ are always positive and there is no x intercept.
- If $a > 1$, the graph of $y = a^x$ has a positive slope and is always increasing, $\frac{d}{dx}(a^x) = a^x \ln a > 0$.
- The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x(\ln a)^2 > 0$.
- In this case a larger value of a gives a steeper curve.
- As $x \to \infty$, $x \ln a$ approaches ∞, since $\ln a > 0$ and therefore $a^x = e^{x \ln a} \to \infty$
- As $x \to -\infty$, $x \ln a$ approaches $-\infty$, since $x < 0$ and $\ln a > 0$. Therefore $a^x = e^{x \ln a} \to 0$.

$$\boxed{\text{For } \; a > 1, \quad \lim_{x \to \infty} a^x = \infty, \quad \lim_{x \to -\infty} a^x = 0}.$$



Functions of the form $(f(x))^{g(x)}$.

Derivatives We now have 4 different types of functions involving bases and powers. So far we have dealt with the first three types:
If a and b are constants and $g(x) > 0$ and $f(x)$ and $g(x)$ are both differentiable functions.

$$\boxed{\frac{d}{dx}a^b = 0, \qquad \frac{d}{dx}(f(x))^b = b(f(x))^{b-1}f'(x), \qquad \frac{d}{dx}a^{g(x)} = g'(x)a^{g(x)}\ln a, \qquad \frac{d}{dx}(f(x))^{g(x)}}$$

For $\frac{d}{dx}(f(x))^{g(x)}$, we use **logarithmic differentiation or write the function as** $(f(x))^{g(x)} = e^{g(x)\ln(f(x))}$ **and use the chain rule**.

Example Differentiate x^{2x^2}, $x > 0$.

Limits

To calculate limits of functions of this type it may help write the function as $(f(x))^{g(x)} = e^{g(x)\ln(f(x))}$.

Example What is $\lim_{x\to\infty} x^{-x}$

General Logarithmic functions

Since $f(x) = a^x$ is a monotonic function whenever $a \neq 1$, it has an inverse which we denote by $f^{-1}(x) = \log_a x$. We get the following from the properties of inverse functions:

$$f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x$$

$$\boxed{\log_a(x) = y \quad \text{if and only if} \quad a^y = x}$$

$$f(f^{-1}(x)) = x \qquad f^{-1}(f(x)) = x$$

$$\boxed{a^{\log_a(x)} = x \qquad \log_a(a^x) = x.}$$

Converting to the natural logarithm

It is not difficult to show that $\log_a x$ has similar properties to $\ln x = \log_e x$. This follows from the **Change of Base Formula** which shows that The function $\log_a x$ is a constant multiple of $\ln x$.

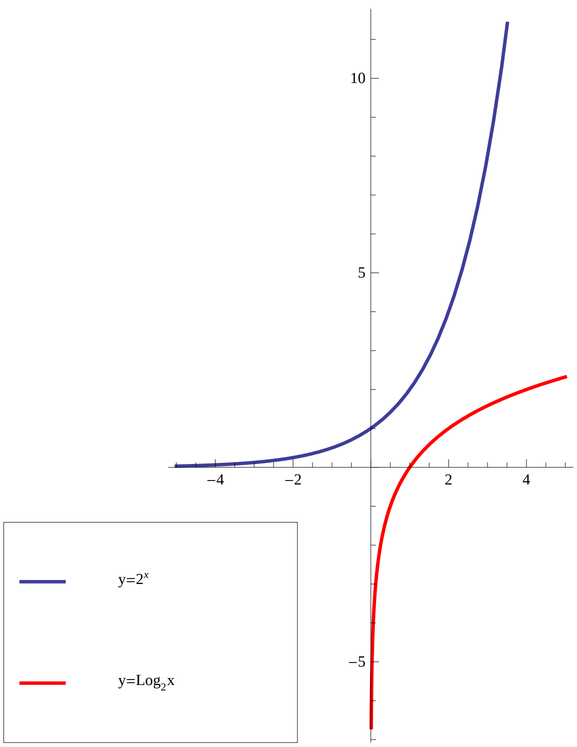
$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

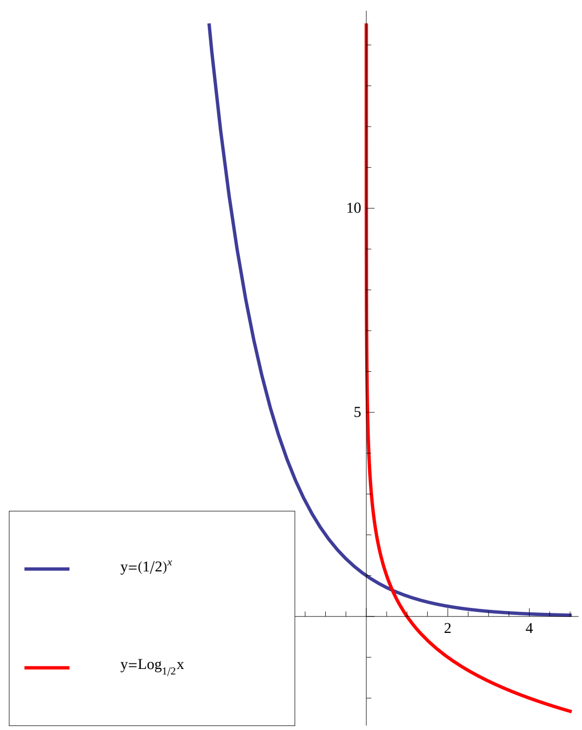
The algebraic properties of the natural logarithm thus extend to general logarithms, by the change of base formula.

$$\log_a 1 = 0, \qquad \log_a(xy) = \log_a(x) + \log_a(y), \qquad \log_a(x^r) = r\log_a(x).$$

for any positive number $a \neq 1$. In fact for most calculations (especially limits, derivatives and integrals) it is advisable to convert $\log_a x$ to natural logarithms. The most commonly used logarithm functions are $\log_{10} x$ and $\ln x = \log_e x$.

Since $\log_a x$ is the inverse function of a^x, it is easy to derive the properties of its graph from the graph $y = a^x$, or alternatively, from the change of base formula $\log_a x = \frac{\ln x}{\ln a}$.





Basic Application

Example Express as a single number $\log_5 25 - \log_5 \sqrt{5}$

Using the change of base formula for Derivatives

From the above change of base formula for $\log_a x$, we can easily derive the following **differentiation formulas**:

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \qquad\qquad \frac{d}{dx}(\log_a g(x)) = \frac{g'(x)}{g(x) \ln a}.}$$

Example Find $\frac{d}{dx} \log_2(x \sin x)$.

A special limit and an approximation of e

We derive the following limit formula by taking the derivative of $f(x) = \ln x$ at $x = 1$:

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \ln(1 + x)^{1/x} = 1.$$

Applying the (continuous) exponential function to the limit we get

$$\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$$

Note If we substitute $y = 1/x$ in the above limit we get

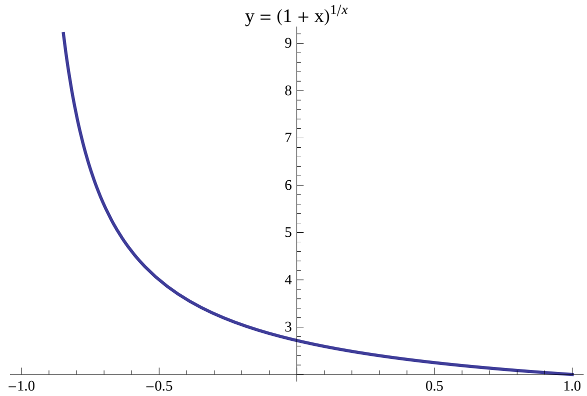
$$\boxed{e = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y \quad \text{and} \quad e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

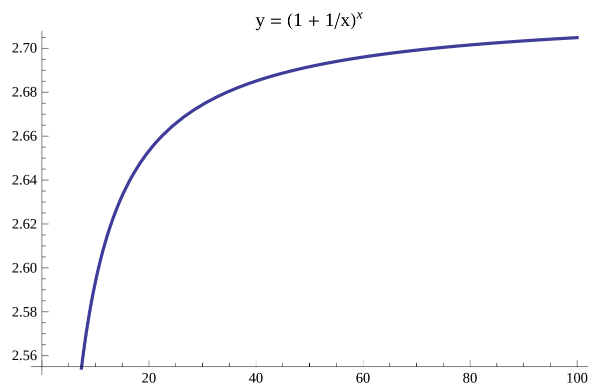
where n is an integer (see graphs below). We look at large values of n below to get an approximation of the value of e.

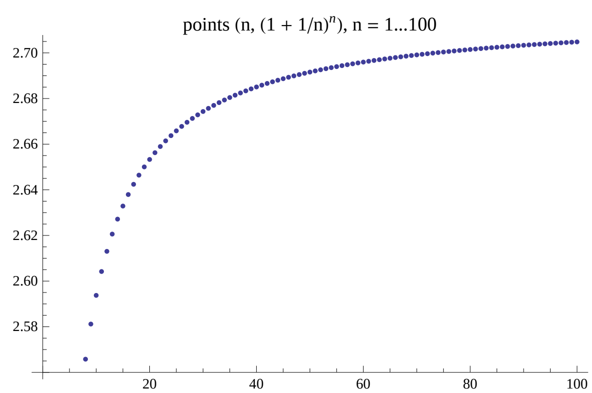
$n = 10 \to \left(1 + \frac{1}{n}\right)^n = 2.59374246, \quad n = 100 \to \left(1 + \frac{1}{n}\right)^n = 2.70481383,$

$n = 100 \to \left(1 + \frac{1}{n}\right)^n = 2.71692393, \quad n = 1000 \to \left(1 + \frac{1}{n}\right)^n = 2.1814593.$

Example Find $\lim_{x \to 0}(1 + \frac{x}{2})^{1/x}$.


y = (1 + x)^{1/x}
9
8
7
6
5
4
3
−1.0
−0.5
0.5
1.0


y = (1 + 1/x)^x
2.70
2.68
2.66
2.64
2.62
2.60
2.58
2.56
20
40
60
80
100


points (n, (1 + 1/n)^n), n = 1...100
2.70
2.68
2.66
2.64
2.62
2.60
2.58
20
40
60
80
100

Extras for discussion at your Friday night Calculus Party

Example Differentiate the following functions:

$$f(x) = 102^x \qquad g(x) = (1000)2^{x^3}, \qquad x^2 + 3^{\sqrt{x}}, \qquad (x^2+3)^{\sqrt{x}}.$$

Example Evaluate the following limits:

$$\lim_{x\to 0} 2^{x^2}, \quad \lim_{x\to 0}(1/2)^{x^2} \quad \lim_{x\to\infty}(x^2 + (1/3)^{\sqrt{x}}),, \quad \lim_{x\to 0}(1+x)^{1/x}, \quad \lim_{x\to 0}(1+\frac{x}{5})^{1/x}$$

Use the change of base formula for the next 3 problems

Example Solve for x if $50 = 2^{x-1}$

Example Evaluate the limit $\lim_{x\to 0}\log_{1/3}(x^2+x)$.

Example Evaluate the integral $\int \frac{1}{x\log_2 x}\,dx$.

$$\lim_{x\to 0}\ln(1+x)^{1/x} = 1.$$

Richter Scale: The Richter scale gives the magnitude of an earthquake to be

$$\log_{10}(I/S)$$

where S =intensity of a standard quake giving an amplitude of 1 micron = 10^{-4} cm on a seismograph 100 km from the epicenter. I = intensity of the earthquake in question measured on a seismograph 100 km from the epicenter (or an estimate thereof from a model).

If a quake has intensity I = 1 (cm on seismograph 100 km from epicenter) what is its magnitude?
If a quake has intensity I = 10 (cm on seismograph 100 km from epicenter) what is its magnitude?
Note that a magnitude 5 quake has an intensity 10 times that of a 4 quake etc....
Chile, 1960, 9.5, Alaska, 1964, 9.2, 2004, Sumatra Indonesia, 9.1. , Had a 3 in Indiana recently ?

Solutions to Extras

proof that $\lim_{x\to 0}(1+x)^{1/x} = e$**:**
Let $f(x) = \ln x$, then

$$f'(1) = \lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\ln(1+h)^{1/h}.$$

Now $f'(x) = 1/x$, therefore $f'(1) = 1$ and

$$\lim_{h\to 0}\ln(1+h)^{1/h} = 1.$$

Applying the exponential (which is a continuous function) to both sides, we get

$$e^{\lim_{h\to 0}\ln(1+h)^{1/h}} = \lim_{h\to 0} e^{\ln(1+h)^{1/h}} = \lim_{h\to 0}(1+h)^h = e^1 = e$$

Example Differentiate the following functions:

$$f(x) = 102^x \qquad g(x) = (1000)2^{x^3}, \quad h(x) = x^2 + 3^{\sqrt{x}}, \quad k(x) = (x^2+3)^{\sqrt{x}}.$$

$$f(x) = 10e^{x\ln 2}, \text{ using chain rule: } f'(x) = 10e^{x\ln 2}\ln 2 = 10(\ln 2)2^x.$$

$$g(x) = (1000)e^{x^3\ln 2}, \text{ using chain rule: } g'(x) = 1000e^{x^3\ln 2}3x^2\ln 2 = 3000x^2(\ln 2)2^{x^3}.$$

$$h(x) = x^2 + e^{\sqrt{x}\ln 3}, \text{ using chain rule: } h'(x) = 2x + e^{\sqrt{x}\ln 3}\frac{1}{2\sqrt{x}}\ln 3 = 2x + \frac{\ln 3}{2\sqrt{x}}3^{\sqrt{x}}.$$

For $y = k(x)$, we can use logarithmic differentiation.

$$y = (x^2+3)^{\sqrt{x}} \quad \to \quad \ln y = \sqrt{x}\ln(x^2+3).$$

Differentiating both sides we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln(x^2+3) + \sqrt{x}\frac{2x}{x^2+3}$$

Multiplying both sides by $y = (x^2+3)^{\sqrt{x}}$, we get

$$\frac{dy}{dx} = \frac{(x^2+3)^{\sqrt{x}}}{2\sqrt{x}}\ln(x^2+3) + \frac{2x^{3/2}(x^2+3)^{\sqrt{x}}}{x^2+3}$$

Example Evaluate the following limits:

$$\lim_{x\to 0} 2^{x^2}, \quad \lim_{x\to 0}\log_2(x^2) \qquad \lim_{x\to\infty}(x^2 + (1/3)^{\sqrt{x}}), \quad \lim_{x\to 0}(1+\frac{x}{5})^{1/x}$$

$$\lim_{x\to 0} 2^{x^2} = 2^{\lim_{x\to 0}(x^2)} = 2^0 = 1.$$

$$\lim_{x\to 0}\log_2(x^2) = \lim_{x\to 0}\frac{\ln(x^2)}{\ln 2} = \frac{\lim_{x\to 0}\ln(x^2)}{\ln 2} = -\infty \text{ since } \ln 2 > 0.$$

$$\lim_{x\to\infty}(x^2 + (1/3)^{\sqrt{x}}) = \lim_{x\to\infty} x^2 + \lim_{x\to\infty}(e)^{\sqrt{x}\ln(1/3)} = \lim_{x\to\infty} x^2 + \lim_{x\to\infty}(e)^{-\sqrt{x}\ln(3)}.$$

As $x \to \infty$, we have $-\sqrt{x}\ln 3 \to -\infty$ and $\lim_{x\to\infty}(e)^{-\sqrt{x}\ln(3)} = 0$. Therefore

$$\lim_{x\to\infty}(x^2 + (1/3)^{\sqrt{x}}) = \lim_{x\to\infty} x^2 = \infty.$$

$$\lim_{x\to 0}(1 + \frac{x}{5})^{1/x} = \lim_{y\to 0}(1+y)^{1/(5y)} = \left[\lim_{y\to 0}(1+y)^{1/(y)}\right]^{1/5} = e^{1/5}, \quad \text{where} \quad y = \frac{x}{5}.$$

Example Solve for x if $50 = 2^{x-1}$

We could apply $\log_2$ to both sides of this equation to get

$$\log_2(50) = \log_2(2^{x-1}) = x - 1.$$

Solving for x, we get $x = \log_2(50) + 1$.
As an alternative option, we could apply ln to both sides of the equation $50 = 2^{x-1}$, to get

$$\ln(50) = \ln(2^{x-1}) = (x-1)\ln 2.$$

Solving for x, we get $x = \frac{\ln(50)}{\ln(2)} + 1$. This is of course the same answer as before.

Example Evaluate the integral $\int \frac{1}{x\log_2 x}\, dx$.

We use the change of base formula to get

$$\int \frac{1}{x\log_2 x}\, dx = \int \frac{\ln(2)}{x\ln(x)}\, dx = \ln(2)\int \frac{1}{x\ln(x)}\, dx.$$

Let $u = \ln(x)$, then $du = \frac{1}{x}\, dx$. We get

$$\ln(2)\int \frac{1}{x\ln(x)}\, dx = \ln(2)\int \frac{1}{u}\, du = \ln(2)\ln(u) + C = \ln(2)\ln(\ln(x)) + C.$$

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