For $a > 0$ and $x$ any real number, we define

$$a^x = e^{x \ln a}, \quad a > 0.$$ \hspace{1cm}

The function $a^x$ is called the exponential function with base $a$.

Note that $\ln(a^x) = x \ln a$ is true for all real numbers $x$ and all $a > 0$. (We saw this before for $x$ a rational number).

**Note:** We have no definition for $a^x$ when $a < 0$, when $x$ is irrational.

For example $2\sqrt{2} = e^{\sqrt{2} \ln 2}$, $2^{-\sqrt{2}}$, $(-2)^{\sqrt{2}}$(no definition).

**Algebraic rules**

The following **Laws of Exponent** follow from the laws of exponents for the natural exponential function.

$$a^{x+y} = a^x a^y \quad a^{x-y} = \frac{a^x}{a^y} \quad (a^x)^y = a^{xy} \quad (ab)^x = a^x b^x$$

**Proof** $a^{x+y} = e^{(x+y) \ln a} = e^{x \ln a + y \ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y$, etc...

**Example** Simplify $\frac{(a^x)^2 a^{x^2+1}}{a^2}$.

**Differentiation**

The following **differentiation rules** also follow from the rules of differentiation for the natural exponential.

$$\frac{d}{dx} (a^x) = \frac{d}{dx} (e^{x \ln a}) = a^x \ln a \quad \frac{d}{dx} (a^{g(x)}) = \frac{d}{dx} e^{g(x) \ln a} = g'(x) a^{g(x) \ln a}$$

**Example** Differentiate the following function:

$$f(x) = (1000)2^{x^2+1}.$$
• Slope: If $0 < a < 1$, the graph of $y = a^x$ has a negative slope and is always decreasing, $\frac{d}{dx}(a^x) = a^x \ln a < 0$. In this case a smaller value of $a$ gives a steeper curve.

• The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x (\ln a)^2 > 0$.

• As $x \to \infty$, $x \ln a$ approaches $-\infty$, since $\ln a < 0$ and therefore $a^x = e^{x \ln a} \to 0$.

• As $x \to -\infty$, $x \ln a$ approaches $\infty$, since both $x$ and $\ln a$ are less than 0. Therefore $a^x = e^{x \ln a} \to \infty$.

For $0 < a < 1$, $\lim_{x \to \infty} a^x = 0$, $\lim_{x \to -\infty} a^x = \infty$.

Graphs of Exponential functions. Case 2: $a > 1$

• y-intercept: The y-intercept is given by $y = a^0 = e^{0 \ln a} = e^0 = 1$.

• x-intercept: The values of $a^x = e^{x \ln a}$ are always positive and there is no x intercept.

• If $a > 1$, the graph of $y = a^x$ has a positive slope and is always increasing, $\frac{d}{dx}(a^x) = a^x \ln a > 0$.

• The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x (\ln a)^2 > 0$.

• In this case a larger value of $a$ gives a steeper curve.

• As $x \to \infty$, $x \ln a$ approaches $\infty$, since $\ln a > 0$ and therefore $a^x = e^{x \ln a} \to \infty$

• As $x \to -\infty$, $x \ln a$ approaches $-\infty$, since $x < 0$ and $\ln a > 0$. Therefore $a^x = e^{x \ln a} \to 0$.

For $a > 1$, $\lim_{x \to \infty} a^x = \infty$, $\lim_{x \to -\infty} a^x = 0$.
Functions of the form \((f(x))^g(x)\).

**Derivatives** We now have 4 different types of functions involving bases and powers. So far we have dealt with the first three types:

If \(a\) and \(b\) are constants and \(g(x) > 0\) and \(f(x)\) and \(g(x)\) are both differentiable functions.

\[
\frac{d}{dx} a^b = 0, \quad \frac{d}{dx} (f(x))^b = b(f(x))^{b-1} f'(x), \quad \frac{d}{dx} a^{g(x)} = g'(x) a^{g(x)} \ln a, \quad \frac{d}{dx} (f(x))^{g(x)}
\]

For \(\frac{d}{dx} (f(x))^g(x)\), we use logarithmic differentiation or write the function as \((f(x))^g(x) = e^{g(x)\ln(f(x))}\) and use the chain rule.

**Example** Differentiate \(x^{2x^2}, x > 0\).

---

**Limits**

To calculate limits of functions of this type it may help write the function as \((f(x))^g(x) = e^{g(x)\ln(f(x))}\).

**Example** What is \(\lim_{x \to \infty} x^{-x}\)

---

**General Logarithmic functions**

Since \(f(x) = a^x\) is a monotonic function whenever \(a \neq 1\), it has an inverse which we denote by \(f^{-1}(x) = \log_a x\). We get the following from the properties of inverse functions:

\[
f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x \\
\log_a(x) = y \quad \text{if and only if} \quad a^y = x \\
f(f^{-1}(x)) = x \quad f^{-1}(f(x)) = x \\
\]

\[
a^{\log_a(x)} = x \quad \log_a(a^x) = x.
\]
Converting to the natural logarithm

It is not difficult to show that \( \log_a x \) has similar properties to \( \ln x = \log_e x \). This follows from the **Change of Base Formula** which shows that The function \( \log_a x \) is a constant multiple of \( \ln x \).

\[
\log_a x = \frac{\ln x}{\ln a}
\]

The algebraic properties of the natural logarithm thus extend to general logarithms, by the change of base formula.

\[
\log_a 1 = 0, \quad \log_a (xy) = \log_a (x) + \log_a (y), \quad \log_a (x^r) = r \log_a (x).
\]

for any positive number \( a \neq 1 \). In fact for most calculations (especially limits, derivatives and integrals) it is advisable to convert \( \log_a x \) to natural logarithms. The most commonly used logarithm functions are \( \log_{10} x \) and \( \ln x = \log_e x \).

Since \( \log_a x \) is the inverse function of \( a^x \), it is easy to derive the properties of its graph from the graph \( y = a^x \), or alternatively, from the change of base formula \( \log_a x = \frac{\ln x}{\ln a} \).

**Basic Application**

**Example** Express as a single number \( \log_5 25 - \log_5 \sqrt{5} \)
Using the change of base formula for Derivatives

From the above change of base formula for \( \log_a x \), we can easily derive the following differentiation formulas:

\[
\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \quad \frac{d}{dx}(\log_a g(x)) = \frac{g'(x)}{g(x) \ln a}.
\]

Example Find \( \frac{d}{dx} \log_2(x \sin x) \).

A special limit and an approximation of \( e \)

We derive the following limit formula by taking the derivative of \( f(x) = \ln x \) at \( x = 1 \):

\[
\lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \ln(1 + x)^{1/x} = 1.
\]

Applying the (continuous) exponential function to the limit we get

\[
e = \lim_{x \to 0} (1 + x)^{1/x}
\]

Note If we substitute \( y = 1/x \) in the above limit we get

\[
e = \lim_{y \to \infty} \left( 1 + \frac{1}{y} \right)^y \quad \text{and} \quad e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n
\]

where \( n \) is an integer (see graphs below). We look at large values of \( n \) below to get an approximation of the value of \( e \).

\[
n = 10 \to \left( 1 + \frac{1}{n} \right)^n = 2.59374246, \quad n = 100 \to \left( 1 + \frac{1}{n} \right)^n = 2.70481383,
\]

\[
n = 100 \to \left( 1 + \frac{1}{n} \right)^n = 2.71692393, \quad n = 1000 \to \left( 1 + \frac{1}{n} \right)^n = 2.1814593.
\]

Example Find \( \lim_{x \to 0} (1 + \frac{x}{2})^{1/x} \).
\[ y = (1 + x)^{1/10} \]

\[ y = (1 + 1/x)^x \]

points \((n, (1 + 1/n)^x), n = 1...100\)
Extras for discussion at your Friday night Calculus Party

Example Differentiate the following functions:

\[ f(x) = 102^x \quad g(x) = (1000)2^{x^3}, \quad x^2 + 3^{\sqrt{x}}, \quad (x^2 + 3)^{\sqrt{x}}. \]

Example Evaluate the following limits:

\[ \lim_{x \to 0} 2^{x^2}, \quad \lim_{x \to 0} (1/2)^{x^2} \quad \lim_{x \to \infty} (x^2 + (1/3)^{\sqrt{x}}), \quad \lim_{x \to 0} (1 + x)^{1/x}, \quad \lim_{x \to 0} (1 + \frac{x}{5})^{1/x}. \]

Use the change of base formula for the next 3 problems

Example Solve for \( x \) if \( 50 = 2^{x - 1} \)

Example Evaluate the limit \( \lim_{x \to 0} \log_{1/3}(x^2 + x) \).

Example Evaluate the integral \( \int \frac{1}{x \log_2 x} \, dx \).

\[ \lim_{x \to 0} \ln(1 + x)^{1/x} = 1. \]

Richter Scale: The Richter scale gives the magnitude of an earthquake to be

\[ \log_{10}(I/S) \]

where \( S \) = intensity of a standard quake giving an amplitude of 1 micron = \( 10^{-4} \) cm on a seismograph 100 km from the epicenter. \( I \) = intensity of the earthquake in question measured on a seismograph 100 km from the epicenter (or an estimate thereof from a model).

If a quake has intensity \( I = 1 \) (cm on seismograph 100 km from epicenter) what is its magnitude?
If a quake has intensity \( I = 10 \) (cm on seismograph 100 km from epicenter) what is its magnitude?
Note that a magnitude 5 quake has an intensity 10 times that of a 4 quake etc....
Chile, 1960, 9.5, Alaska, 1964, 9.2, 2004, Sumatra Indonesia, 9.1. , Had a 3 in Indiana recently ?
Solutions to Extras

**proof that** \( \lim_{x \to 0} (1 + x)^{1/x} = e \):

Let \( f(x) = \ln x \), then

\[
f'(1) = \lim_{h \to 0} \frac{\ln(1 + h) - \ln 1}{h} = \lim_{h \to 0} \frac{\ln(1 + h)}{h} = \lim_{h \to 0} (1 + h)^{1/h}.
\]

Now \( f'(x) = 1/x \), therefore \( f'(1) = 1 \) and

\[
\lim_{h \to 0} \ln(1 + h)^{1/h} = 1.
\]

Applying the exponential (which is a continuous function) to both sides, we get

\[
e^{\lim_{h \to 0} \ln(1 + h)^{1/h}} = e^{\lim_{h \to 0} (1 + h)^{1/h}} = e.
\]

**Example** Differentiate the following functions:

\[
f(x) = 102^x \quad g(x) = (1000)2^{x^3}, \quad h(x) = x^2 + 3\sqrt{x}, \quad k(x) = (x^2 + 3)^\sqrt{x}.
\]

\[
f(x) = 10e^{x\ln 2}, \quad \text{using chain rule: } f'(x) = 10e^{x\ln 2} \ln 2 = 10(\ln 2)2^x.
\]

\[
g(x) = (1000)e^{x\ln 2}, \quad \text{using chain rule: } g'(x) = 1000e^{x\ln 2}3x^2 \ln 2 = 3000x^2(\ln 2)2^{x^3}.
\]

\[
h(x) = x^2 + e^{\sqrt{x}\ln 3}, \quad \text{using chain rule: } h'(x) = 2x + e^{\sqrt{x}\ln 3} \frac{1}{2\sqrt{x}} \ln 3 = 2x + \frac{\ln 3}{2\sqrt{x}3^{\sqrt{x}}}
\]

For \( y = k(x) \), we can use logarithmic differentiation.

\[
y = (x^2 + 3)^\sqrt{x} \quad \rightarrow \quad \ln y = \sqrt{x} \ln(x^2 + 3).
\]

Differentiating both sides we get

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \ln(x^2 + 3) + \sqrt{x} \frac{2x}{x^2 + 3}
\]

Multiplying both sides by \( y = (x^2 + 3)^\sqrt{x} \), we get

\[
\frac{dy}{dx} = \frac{(x^2 + 3)^\sqrt{x}}{2\sqrt{x}} \ln(x^2 + 3) + \frac{2x^3/2(x^2 + 3)^\sqrt{x}}{x^2 + 3}
\]

**Example** Evaluate the following limits:

\[
\lim_{x \to 0} 2^{x^2}, \quad \lim_{x \to 0} \log_2(x^2) \quad \lim_{x \to \infty} (x^2 + (1/3)^{\sqrt{x}}), \quad \lim_{x \to 0} (1 + \frac{x}{5})^{1/x}
\]

\[
\lim_{x \to 0} 2^{x^2} = 2^{\lim_{x \to 0} (x^2)} = 2^0 = 1.
\]

\[
\lim_{x \to 0} \log_2(x^2) = \lim_{x \to 0} \frac{\ln(x^2)}{\ln 2} = \lim_{x \to 0} \frac{\ln(x^2)}{\ln 2} = -\infty \text{ since } \ln 2 > 0.
\]

\[
\lim_{x \to \infty} (x^2 + (1/3)^{\sqrt{x}}) = \lim_{x \to \infty} x^2 + \lim_{x \to \infty} (e)^{\sqrt{x}\ln(1/3)} = \lim_{x \to \infty} x^2 + \lim_{x \to \infty} (e)^{-\sqrt{x}\ln(3)}.
\]
As \( x \to \infty \), we have \(-\sqrt{x} \ln 3 \to -\infty \) and \( \lim_{x \to \infty} (e^{-\sqrt{x} \ln(3)}) = 0 \). Therefore

\[
\lim_{x \to \infty} (x^2 + (1/3)\sqrt{x}) = \lim_{x \to \infty} x^2 = \infty.
\]

\[
\lim_{x \to 0} (1 + \frac{x}{5})^{1/x} = \lim_{y \to 0} (1 + y)^{1/(5y)} = \left[ \lim_{y \to 0} (1 + y)^{1/(y)} \right]^{1/5} = e^{1/5}, \text{ where } y = \frac{x}{5}.
\]

**Example** Solve for \( x \) if \( 50 = 2^{x-1} \)

We could apply \( \log_2 \) to both sides of this equation to get

\[
\log_2(50) = \log_2(2^{x-1}) = x - 1.
\]

Solving for \( x \), we get \( x = \log_2(50) + 1 \).

As an alternative option, we could apply \( \ln \) to both sides of the equation \( 50 = 2^{x-1} \), to get

\[
\ln(50) = \ln(2^{x-1}) = (x - 1) \ln 2.
\]

Solving for \( x \), we get \( x = \frac{\ln(50)}{\ln(2)} + 1 \). This is of course the same answer as before.

**Example** Evaluate the integral \( \int \frac{1}{x \log_2 x} \, dx \).

We use the change of base formula to get

\[
\int \frac{1}{x \log_2 x} \, dx = \int \frac{\ln(2)}{x \ln(x)} \, dx = \ln(2) \int \frac{1}{x \ln(x)} \, dx.
\]

Let \( u = \ln(x) \), then \( du = \frac{1}{x} \, dx \). We get

\[
\ln(2) \int \frac{1}{x \ln(x)} \, dx = \ln(2) \ln(\ln(x)) + C = \ln(2) \ln(\ln(x)) + C.
\]

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\[
\log_{10}(I/S)
\]

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