Lecture 7 : Indeterminate Forms

Recall that we calculated the following limit using geometry in Calculus 1:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Definition An indeterminate form of the type $\frac{0}{0}$ is a limit of a quotient where both numerator and denominator approach 0.

Example

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} \qquad \lim_{x \to \infty} \frac{x^{-2}}{e^{-x}} \qquad \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

Definition An indeterminate form of the type $\frac{\infty}{\infty}$ is a limit of a quotient $\frac{f(x)}{g(x)}$ where $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$.

Example

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x} \qquad \lim_{x \to 0^+} \frac{x^{-1}}{\ln x}.$$

L'Hospital's Rule Suppose lim stands for any one of

$$\lim_{x\to a} \qquad \lim_{x\to a^+} \qquad \lim_{x\to a^-} \qquad \lim_{x\to\infty} \qquad \lim_{x\to-\infty}$$

and $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If $lim \frac{f'(x)}{g'(x)}$ is a finite number L or is $\pm\infty$, then

$$\boxed{lim \frac{f(x)}{g(x)} = lim \frac{f'(x)}{g'(x)}.}$$

(Assuming that $f(x)$ and $g(x)$ are both differentiable in some open interval around a or ∞ (as appropriate) except possible at a, and that $g'(x) \neq 0$ in that interval).

Definition $lim\ f(x)g(x)$ is **an indeterminate form of the type** $0 \cdot \infty$ if

$$lim\ f(x) = 0 \quad \text{and} \quad lim\ g(x) = \pm\infty.$$

Example $\lim_{x\to\infty} x\tan(1/x)$

We can convert the above indeterminate form to an indeterminate form of type $\frac{0}{0}$ by writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)}$$

or to an indeterminate form of the type $\frac{\infty}{\infty}$ by writing

$$f(x)g(x) = \frac{g(x)}{1/f(x)}.$$

We them apply L'Hospital's rule to the limit.

Indeterminate Forms of the type 0^0, ∞^0, 1^∞.

Type	Limit		
0^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 0$	$lim\ g(x) = 0$
∞^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = \infty$	$lim\ g(x) = 0$
1^∞	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 1$	$lim\ g(x) = \infty$

Example $\lim_{x\to 0}(1+x^2)^{\frac{1}{x}}$.

Method

1. Look at $lim\ \ln[f(x)]^{g(x)} = lim\ g(x)\ln[f(x)]$.

2. Use L'Hospital to find $lim\ g(x)\ln[f(x)] = \alpha$. ($\alpha$ might be finite or $\pm\infty$ here.)

3. Then $lim f(x)^{g(x)} = lim\ e^{\ln[f(x)]^{g(x)}} = e^\alpha$ since e^x is a continuous function. (where e^∞ should be interpreted as ∞ and $e^{-\infty}$ should be interpreted as 0.)

Indeterminate Forms of the type $\infty - \infty$ occur when we encounter a limit of the form $lim(f(x) - g(x))$ where $lim\ f(x) = lim\ g(x) = \infty$ or $lim\ f(x) = lim\ g(x) = -\infty$

Example $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x}$

To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...

Try these Extra Fun Examples over Lunch

$$\lim_{x \to -\infty} \frac{2^x}{\sin(\frac{1}{x})}$$

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x}$$

$$\lim_{x \to \infty} x \tan(1/x)$$

$$\lim_{x \to 0^+} (e^{2x} - 1)^{\frac{1}{\ln x}}$$

$$\lim_{x \to 1} (x)^{\frac{1}{x-1}}$$

Lecture 7 : Indeterminate Forms

$$\lim_{x\to-\infty} \frac{2^x}{\sin(\frac{1}{x})}$$

(Note: You could use the sandwich theorem from Calc 1 for this if you prefer.)
This is an indeterminate form of type $\frac{0}{0}$. By L'Hospitals rule it equals:

$$\lim_{x\to-\infty} \frac{(\ln 2)2^x}{-\frac{1}{x^2}\cos(\frac{1}{x})} = \lim_{x\to-\infty} \frac{\ln 2}{\cos(\frac{1}{x})} \lim_{x\to-\infty} \frac{2^x}{-\frac{1}{x^2}}$$

$$= (\ln 2) \lim_{x\to-\infty} \frac{-x^2}{2^{-x}}$$

Applying L'Hospital again, we get that this equals

$$(\ln 2) \lim_{x\to-\infty} \frac{-2x}{-(\ln 2)2^{-x}}$$

Applying L'Hospital a third time, we get that this equals

$$\frac{2(\ln 2)}{\ln 2} \lim_{x\to-\infty} \frac{1}{-(\ln 2)2^{-x}} = 0$$

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

This is an indeterminate form of type $\frac{\infty}{\infty}$
Applying L'Hospital's rule we get that it equals

$$\lim_{x\to 0^+} \frac{1/x}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x} \lim_{x\to 0^+} \frac{1}{\cos x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x}$$

We can apply L'Hospital's rule again to get that the above limit equals

$$\lim_{x\to 0^+} \frac{-2\sin x \cos x}{1} = 0$$

$$\lim_{x\to\infty} x \tan(1/x)$$

Rearranging this, we get an indeterminate form of type $\frac{0}{0}$

$$\lim_{x\to\infty} \frac{\tan(1/x)}{1/x} = \lim_{x\to\infty} \frac{\frac{-1}{x^2}\sec^2(1/x)}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{\cos^2(1/x)} = 1$$

$$\lim_{x\to 0^+}(e^{2x}-1)^{\frac{1}{\ln x}}$$

is an indeterminate form of type 0^0. Using continuity of the exponential function, we get $\lim_{x\to 0^+}(e^{2x}-1)^{\frac{1}{\ln x}} = e^{\lim_{x\to 0^+}\ln((e^{2x}-1)^{\frac{1}{\ln x}})} = e^{\lim_{x\to 0^+}\frac{1}{\ln x}\ln(e^{2x}-1)}$
For

$$\lim_{x\to 0^+}\frac{\ln(e^{2x}-1)}{\ln x}$$

we apply L'Hospital to get:

$$\lim_{x\to 0^+}\frac{\ln(e^{2x}-1)}{\ln x} = \lim_{x\to 0^+}\frac{\frac{(2e^{2x})}{e^{2x}-1}}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{(2xe^{2x})}{e^{2x}-1}$$

We apply L'Hospital again to get :

$$= \lim_{x\to 0^+}\frac{(2(e^{2x}+2xe^{2x}))}{2e^{2x}} = 1$$

Substiuting this into the original limit, we get

$$\lim_{x\to 0^+}(e^{2x}-1)^{\frac{1}{\ln x}} = e^1 = e$$

$$\lim_{x\to 1}(x)^{\frac{1}{x-1}}$$

$$\lim_{x\to 1}(x)^{\frac{1}{x-1}} = e^{\lim_{x\to 1}\ln(x^{\frac{1}{x-1}})} = e^{\lim_{x\to 1}\ln(x^{\frac{1}{x-1}})}$$

Focusing on the power we get

$$\lim_{x\to 1}\ln(x^{\frac{1}{x-1}}) = \lim_{x\to 1}\frac{\ln x}{x-1}$$

This is an indeterminate form of type $\frac{0}{0}$ so we can apply L'Hospital's rule to get

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{1/x}{1} = 1$$

Substitution this for the power of e above we get

$$\lim_{x\to 1}(x)^{\frac{1}{x-1}} = e^1 = e$$

$$\lim_{x\to\infty}\sqrt{x^2+\ln x}-x$$

This is an indeterminate form of type $\infty-\infty$

$$= \lim_{x\to\infty}(\sqrt{x^2+\ln x}-x)\frac{\sqrt{x^2+\ln x}+x}{\sqrt{x^2+\ln x}+x} = \lim_{x\to\infty}\frac{x^2+\ln x-x^2}{\sqrt{x^2+\ln x}+x}$$

$$= \lim_{x\to\infty}\frac{\ln x}{\sqrt{x^2+\ln x}+x}$$

This in an indeterminate form of type $\frac{\infty}{\infty}$. We can apply L'Hospital to get

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x^2+\ln x}+x} = \lim_{x\to\infty} \frac{1/x}{\frac{2x+1/x}{2\sqrt{x^2+\ln x}}+1}$$

We calculate

$$\lim_{x\to\infty} \frac{2x+1/x}{2\sqrt{x^2+\ln x}}$$

by dividing the numerator and denominator by x to get

$$\lim_{x\to\infty} \frac{2x+1/x}{2\sqrt{x^2+\ln x}} = \lim_{x\to\infty} \frac{2+1/x^2}{2\sqrt{1+\frac{\ln x}{x^2}}}$$

Applying L'Hospital to get $\lim \ln x/x^2 = \lim 1/2x^2 = 0$, we get

$$\lim_{x\to\infty} \frac{2x+1/x}{2\sqrt{x^2+\ln x}} = 1$$

and using this, we get

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x^2+\ln x}+x} = \lim_{x\to\infty} \frac{1/x}{\frac{2x+1/x}{2\sqrt{x^2+\ln x}}+1} = \frac{0}{2} = 0$$

Now this gives

$$\boxed{\lim_{x\to\infty} \sqrt{x^2+\ln x}-x = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x^2+\ln x}+x} = 0}$$