

Essential Background

A **Polynomial** $P(x)$ is a linear sum of powers of x , for example $3x^3 + 3x^2 + x + 1$ or $x^5 - x$.

The **degree** of a polynomial $P(x)$ is the highest power occurring in the polynomial, for example the degree of $3x^3 + 3x^2 + x + 1$ is 3 and the degree of $x^5 - x$ is 5.

Fundamental Theorem of Algebra Every polynomial can be factored into linear factors of the form $ax + b$ and irreducible quadratic factors $(ax^2 + bx + c$ where $b^2 - 4ac < 0$) where a, b and c are constants.

For example $3x^3 + 3x^2 + x + 1 = (x + 1)(3x^2 + 1)$ and $x^5 - x = x(x^4 - 1) = x(x^2 - 1)(x^2 + 1) = x(x - 1)(x + 1)(x^2 + 1)$.

A **Rational Function** is a quotient of 2 polynomials $\frac{P(x)}{Q(x)}$.

Partial Fraction Decomposition

The rational function $\frac{R(x)}{Q(x)}$ is a **proper rational function** is $\text{Deg}R(x) < \text{Deg}Q(x)$. In this case, we can write the rational function as a sum of **Partial Fractions** of the form

$$\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^i}$$

where A and B are constants and i is a non-negative integer.

We already know how to integrate these partial fractions using substitution, trigonometric substitution or logarithms. We will go through the method of solving for the constants in the partial fraction expansion of a proper rational function in steps.

Example How would we start to integrate the following :

$$(a) \int \frac{5}{x+1} dx \quad (b) \int \frac{2x+1}{x^2+1} dx \quad (c) \int \frac{1}{(x^2+2x+3)^2} dx.$$

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- ▶ (a) Substitute $u = x + 1$
- (b) $\int \frac{2x+1}{x^2+1} dx = \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx$
- (c) Complete the square and use trig substitution.

Partial Fraction Decomposition, Distinct Linear Factors

Step 1 : The Denominator $Q(x)$ is a product of distinct linear factors

If $Q(x) = (a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)$ we include a quotient of the form $\frac{A_i}{(a_ix + b_i)}$ for each term in the partial fraction expansion. We write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \dots + \frac{A_n}{(a_nx + b_n)}$$

and solve for A_1, A_2, \dots, A_n by multiplying this equation by the lowest common denominator of the Right Hand Side which is the product of the linear factors $Q(x)$.

Note: we know how to evaluate the integral $\int \frac{A_i}{(a_ix + b_i)} dx$ using a substitution and logarithms.

Example Evaluate

$$\int \frac{1}{x^2 - 25} dx$$

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► Here $P(x) = 1$ and $Q(x) = x^2 - 25 = (x - 5)(x + 5)$.

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Partial fraction Decomp. Distinct Linear Factors

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Partial fraction Decomp. Distinct Linear Factors

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- ▶ Multiply both sides by the lowest common denominator of R.H.S. $(= (x - 5)(x + 5))$ to get

$$\frac{(x - 5)(x + 5)}{x^2 - 25} = \frac{A(x - 5)(x + 5)}{(x - 5)} + \frac{B(x - 5)(x + 5)}{(x + 5)}$$

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$$1 = A(x + 5) + B(x - 5)$$

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We solve for A and B by comparing co-efficients

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- ▶ $1 = A(x + 5) + B(x - 5)$
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- ▶ $A + B = 0 \rightarrow -A = B$. Using this in $5(A - B) = 1 \rightarrow 5(2A) = 1 \rightarrow A = \frac{1}{10}$ and $B = -\frac{1}{10}$.

Partial fraction Decomp. Distinct Linear Factors

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- ▶ Thus $\frac{1}{x^2-25} = \frac{(1/10)}{x-5} - \frac{(1/10)}{x+5}$. (check)

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- ▶ Thus $\frac{1}{x^2-25} = \frac{(1/10)}{x-5} - \frac{(1/10)}{x+5}$. (check)
- ▶ $\int \frac{1}{x^2-25} dx = \frac{1}{10} \int \frac{1}{x-5} dx - \frac{1}{10} \int \frac{1}{x+5} dx$
- ▶ Using substitutions $u = x - 5$ and $w = x + 5$, we get
$$\begin{aligned} \int \frac{1}{x^2-25} dx &= \frac{1}{10} \int \frac{1}{u} du - \frac{1}{10} \int \frac{1}{w} dw \\ &= \frac{1}{10} \ln |u| - \frac{1}{10} \ln |w| + C = \frac{1}{10} \ln |x - 5| - \frac{1}{10} \ln |x + 5| + C \\ &= \frac{1}{10} \ln \left| \frac{x-5}{x+5} \right| + C. \end{aligned}$$

Partial fraction Decomp. Repeated Linear Factors

Step 2 The denominator has repeated linear factors, that is factors of the form $(a_ix + b_i)^k$ where $k > 1$.

For every factor of type $(a_ix + b_i)^k$ in the denominator we include a sum of type

$$\frac{A_1}{(a_ix + b_i)} + \frac{A_2}{(a_ix + b_i)^2} + \cdots + \frac{A_n}{(a_ix + b_i)^k}$$

in the partial fractions decomposition of the rational function.

For Example, the partial fractions expansion of

$$\frac{x^3 + 2x + 2}{(x - 2)^3(x - 1)^2}$$

looks like

$$\frac{A_1}{x - 2} + \frac{A_2}{(x - 2)^2} + \frac{A_3}{(x - 2)^3} + \frac{B_1}{x - 1} + \frac{B_2}{(x - 1)^2}$$

Note that we can integrate all of these partial fractions using logarithms or integration of powers.

Partial fraction Decomp. Repeated Linear Factors

Example Evaluate

$$\int \frac{2x + 4}{x^3 - 2x^2} dx$$

Partial fraction Decomp. Repeated Linear Factors

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- $\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$

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- $\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$
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($= x^2(x-2)$) to get $\frac{(2x+4)x^2(x-2)}{x^2(x-2)} = \frac{Ax^2(x-2)}{x} + \frac{Bx^2(x-2)}{x^2} + \frac{Cx^2(x-2)}{x-2}$

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- ▶ $2x+4 = Ax(x-2) + B(x-2) + Cx^2$

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- ▶ $2x+4 = Ax(x-2) + B(x-2) + Cx^2$
- ▶ $2x+4 = Ax^2 - 2Ax + Bx - 2B + Cx^2 = (A+C)x^2 + (B-2A)x - 2B$

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- ▶ $2x+4 = Ax(x-2) + B(x-2) + Cx^2$
- ▶ $2x+4 = Ax^2 - 2Ax + Bx - 2B + Cx^2 = (A+C)x^2 + (B-2A)x - 2B$
- ▶ We must have $A+C=0$ and $B-2A=2$ and $-2B=4$.

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- ▶ Multiply both sides by the lowest common denominator of R.H.S.
($= x^2(x-2)$) to get $\frac{(2x+4)x^2(x-2)}{x^2(x-2)} = \frac{Ax^2(x-2)}{x} + \frac{Bx^2(x-2)}{x^2} + \frac{Cx^2(x-2)}{x-2}$
- ▶ $2x+4 = Ax(x-2) + B(x-2) + Cx^2$
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- ▶ We must have $A+C=0$ and $B-2A=2$ and $-2B=4$.
- ▶ $-2B=4 \rightarrow \boxed{B=-2}$. Using this in
 $B-2A=2 \rightarrow -2A=4 \rightarrow \boxed{A=-2}$. Now $A+C=0 \rightarrow \boxed{C=2}$.

Partial fraction Decomp. Repeated Linear Factors

Example Evaluate $\int \frac{2x+4}{x^3-2x^2} dx$.

- We write out the partial fractions decomposition of $\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$. For every factor of type $(a_i x + b_i)^k$ in the denominator we include a sum of type

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$$B - 2A = 2 \rightarrow -2A = 4 \rightarrow \boxed{A = -2}. \text{ Now } A + C = 0 \rightarrow \boxed{C = 2}.$$

► Thus $\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$ (check)

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► $= -2 \ln |x| + 2x^{-1} + 2 \ln |x-2| + C.$

Partial fraction Decomp. Distinct Quadratic Factors

Step 3: The denominator $Q(x)$ has factors which are irreducible quadratics, none of which are repeated,

that is factors of the form $a_ix^2 + b_ix + c_i$ where $b_i^2 - 4a_ic_i < 0$.

In this case we include a term of the form

$$\frac{A_ix + B_i}{a_ix^2 + b_ix + c_i}$$

in the partial fractions decomposition for each such factor. Note that we can integrate this using a combination of substitution and the fact that

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C.$$

Example Evaluate

$$\int \frac{(x^2 + x + 1)}{x(x^2 + 1)} dx$$

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