Midpoint Approximation

Sometimes, we need to approximate an integral of the form \( \int_{a}^{b} f(x) \, dx \) and we cannot find an antiderivative in order to evaluate the integral. Also we may need to evaluate \( \int_{a}^{b} f(x) \, dx \) where we do not have a formula for \( f(x) \) but we have data describing a set of values of the function.

**Review**

We might approximate the given integral using a Riemann sum. Already we have looked at the left end-point approximation and the right end point approximation to \( \int_{a}^{b} f(x) \, dx \) in Calculus 1. We also looked at **the midpoint approximation** \( M \):

**Midpoint Rule** If \( f \) is integrable on \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i) \Delta x = \Delta x (f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)),
\]

where

\[
\Delta x = \frac{b - a}{n} \quad \text{and} \quad x_i = a + i \Delta x \quad \text{and} \quad \bar{x}_i = \frac{1}{2} (x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i].
\]
**Example** Use the midpoint rule with \( n = 6 \) to approximate \( \int_1^4 \frac{1}{x} \, dx \).

\( (= \ln(4) = 1.386294361) \)

Fill in the tables below:
Example Use the midpoint rule with $n = 6$ to approximate $\int_{1}^{4} \frac{1}{x} \, dx$.

$(\approx \ln(4) = 1.386294361)$

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\[ \bar{x}_i = \frac{1}{2}(x_i - 1 + x_i) \]

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( x_0 = 1 )</th>
<th>( x_1 = 3/2 )</th>
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\[ M_6 = P_6 \sum f(\bar{x}_i) \Delta x = \frac{1}{2} \left( \frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \frac{4}{13} + \frac{4}{15} \right) \equiv 1.376934177 \]
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We can also approximate a definite integral \( \int_a^b f(x) \, dx \) using an approximation by trapezoids as shown in the picture below for \( f(x) \geq 0 \)

The area of the trapezoid above the interval \([x_i, x_{i+1}]\) is \( \Delta x \left[ \frac{f(x_i) + f(x_{i+1})}{2} \right] \).

**Trapezoidal Rule** If \( f \) is integrable on \([a, b]\), then

\[
\int_a^b f(x) \, dx \approx T_n = \frac{\Delta x}{2} \left( f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right)
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\[ f(x_i) = \frac{1}{x_i} \quad \begin{array}{cccccccc}
  x_0 & 1 & x_1 & 3/2 & x_2 & 2 & x_3 & 5/2 & x_4 & 3 & x_5 & 7/2 & x_6 & 4 \\
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\[x_i\]

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\]

\[
= \frac{1}{4} \left( 1 + 2 \left( \frac{2}{3} \right) + 2 \left( \frac{1}{2} \right) + 2 \left( \frac{2}{5} \right) + 2 \left( \frac{1}{3} \right) + 2 \left( \frac{2}{7} \right) + \frac{1}{4} \right)
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\[ = 1.405357143. \]
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- The error for the midpoint approximation above is

\[ E_M = \int_1^4 \frac{1}{x} \, dx - M_6 = 1.386294361 - 1.376934177 = 0.00936018 \]

- The error for the trapezoidal approximation above is

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- **Error Bounds**  If \(|f''(x)| \leq K\) for \(a \leq x \leq b\). Let \(E_T\) and \(E_M\) denote the errors for the trapezoidal approximation and midpoint approximation respectively, then

  \[
  |E_T| \leq \frac{K(b - a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b - a)^3}{24n^2}
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Give an upper bound for the error in the trapezoidal approximation of $\int_1^4 \frac{1}{x} \, dx$ when $n = 10$. 

Note that the bound for the error given by the formula is conservative since it turns out to give $|E_T| \leq 0.045$ when $n = 10$, compared to a true error of $|E_T| = 0.00696667$. 
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Annette Pilkington  Approaching an integral
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|T_{10} - \int_1^4 \frac{1}{x} \, dx| = |E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(4-1)^3}{12(10)^2} = 0.045
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Error of Approximation

\[ |E_T| \leq \frac{K(b - a)^3}{12n^2} \text{ and } |E_M| \leq \frac{K(b - a)^3}{24n^2} \]

**Example** (b) Give an upper bound for the error in the midpoint approximation of \( \int_1^4 \frac{1}{x} \, dx \) when \( n = 10 \).

(c) Using the error bounds given above determine how large should \( n \) be to ensure that the trapezoidal approximation is accurate to within 0.000001 = 10^{-6}?
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- As above, we can use \( K = 2 \) to get

\[ |E_M| \leq \frac{2(b - a)^3}{24n^2} = \frac{2(3)^3}{24(10)^2} = 0.0225. \]

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- We want \( |E_T| \leq 10^{-6} \).
Midpoint Approximation Trapezoidal Rule Error Simpson’s Rule

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\[ |E_T| \leq \frac{K(b - a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b - a)^3}{24n^2} \]

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▷ As above, we can use \( K = 2 \) to get

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- or \( n \geq \sqrt{\frac{(10^6)2(27)}{12}} = 2121.32 \), \( n = 2122 \) will work.
**Simpson’s Rule**

We can also approximate a definite integral using parabolas to approximate the curve as in the picture below. [note n is even].

Three points determine a unique parabola. We draw a parabolic segment using the three points on the curve above $x_0, x_1, x_2$. We draw a second parabolic segment using the three points on the curve above $x_2, x_3, x_4$ etc... The area of the parabolic region beneath the parabola above the interval $[x_{i-1}, x_{i+1}]$ is $\frac{\Delta x}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})]$. We estimate the integral by summing the areas of the regions below these parabolic segments to get **Simpson’s Rule** for even $n$:

$$
\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))
$$

where

$$
\Delta x = \frac{b - a}{n} \quad \text{and} \quad x_i = a + i\Delta x \quad \text{and}.
$$

In fact we have $S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$. 

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**Annette Pilkington**

**Approximating an integral**
Simpson’s Rule

\[ \int_{a}^{b} f(x) \, dx \approx S_n = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)) \]

**Example** Use Simpson’s rule with \( n = 6 \) to approximate \( \int_{1}^{4} \frac{1}{x} \, dx \). (= ln(4) = 1.386294361)
Fill in the tables below:
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\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))
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- The error in this estimate is \( E_S = \int_1^4 \frac{1}{x} dx - S_6 = \)

\[ 1.386294361 - 1.387698413 = -0.00140405 \]
Error Bound Simpson’s Rule

**Error Bound for Simpson’s Rule** Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If $E_S$ is the error involved in using Simpson’s Rule, then

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