Moments and Center of Mass

If we have masses $m_1, m_2, \ldots, m_n$ at points $x_1, x_2, \ldots, x_n$ along the x-axis, the **moment of the system around the origin** is

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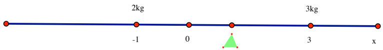

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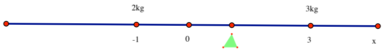

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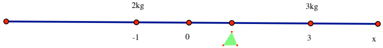

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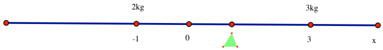
- $\bar{x} = \frac{M_0}{m} = \frac{7}{m_1+m_2} = \frac{7}{2+3} = \frac{7}{5}$ (see the balance point on the diagram)

Law of Archimedes

By a **Law of Archimedes** if we have masses m_1 and m_2 on a rod (of negligible mass) on opposite sides of a fulcrum, at distances d_1 and d_2 from the fulcrum, the rod will balance if $m_1 d_1 = m_2 d_2$. (or in general if we place masses $m_1, m_2, \ldots, m_n$ at distances $d_1, d_2, \ldots d_n$ from the fulcrum the rod will balance if the center of mass is at the fulcrum.)



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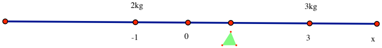

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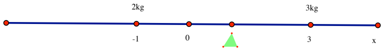

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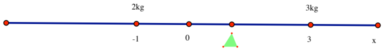

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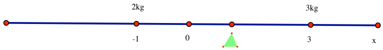

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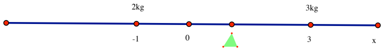

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- That is if we place a mass of $2 + 3 = 5$kg at $\bar{x} = \frac{7}{5}$, the new system has moment $M_0 = 5 \cdot \frac{7}{5} = 7$ and $\bar{x} = \frac{M_0}{5}$.

Two dimensional system

For a two dimensional system, we use x and y axes for reference. We now have moments about each axis. If we have a system with masses $m_1, m_2, \ldots, m_n$ at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ respectively, then the moment about the y axis is given by

$$\boxed{M_y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}$$

and the moment about the x axis is given by

$$\boxed{M_x = m_1 y_1 + m_2 y_2 + \cdots + m_n y_n.}$$

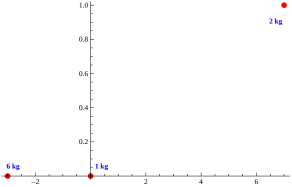
These moments measure the tendency of the system to rotate about the x and y axes respectively.
The **Center of Mass** of the system is given by $(\bar{x}, \bar{y})$ where

$$\boxed{\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} \quad \text{for} \quad m = m_1 + m_2 + \cdots + m_n.}$$

2-D system Example

Example Find the moments and center of mass of a system of objects that have masses

kg	2	1	6
position	$(7,1)$	$(0,0)$	$(-3,0)$



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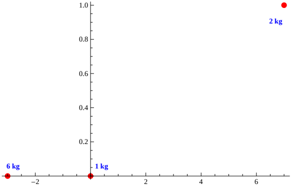

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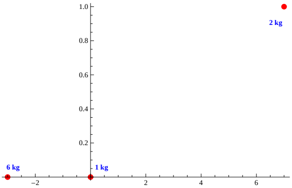

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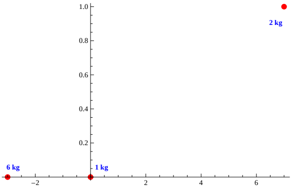

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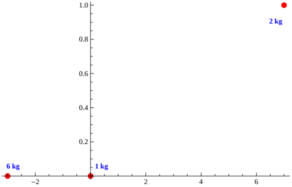

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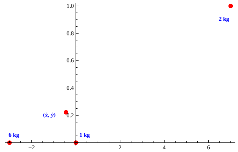

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- $\bar{x} = \frac{M_y}{m} = \frac{-4}{9}, \quad \bar{y} = \frac{M_x}{m} = \frac{2}{9}.$
- Center of Mass $= (\bar{x}, \bar{y}) = (\frac{-4}{9}, \frac{2}{9}).$

Note that **a system with all of the mass placed at the center of mass, has the same moments as the original system.**

- A system with a mass of 9 kg placed at the point $(\frac{-4}{9}, \frac{2}{9})$ has

$$M_y = \frac{9(\,-4/9)}{9} = -4, \quad M_x = \frac{9(2/9)}{9} = 2, \quad (\bar{x}, \bar{y}) = (\frac{-4}{9}, \frac{2}{9}).$$

2-D systems Plates with symmetry

If we have a a thin plate (which occupies a region $\mathfrak{R}$ of the plane) with uniform density ρ, we are interested in calculating its moments about the x and y axis (M_x and M_y respectively) and its center of mass or **centroid**. These are defined in a way that agrees with our intuition. A line of symmetry of the plate (or region) is a line for which a 180^o rotation of the plate about the line makes no change in the plate's appearance. **The Centroid or center of mass lies on each line of symmetry of the plate**. Hence if we have 2 different lines of symmetry, the centroid must be at their intersection.

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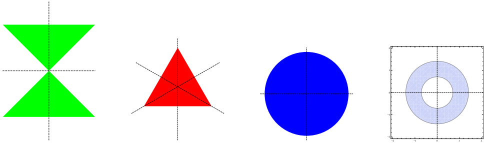

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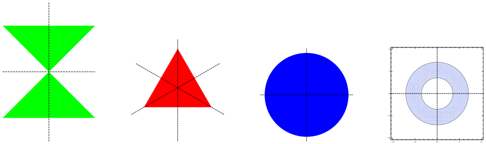

- Note that the center of mass does not have to be in the region (check the figure on the right).

2-D systems regions under curves

The **moments** of the plate or region are defined so that if all of the mass of the plate is centered at the centroid, the system has the same moments. Also the moments of the union of two non-overlapping regions should be the sum of the moments of the individual regions.

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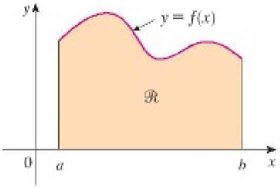
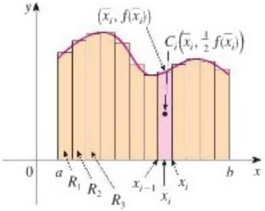
- These two intuitive principles allow us to use both symmetry and Riemann sums to find formulas for the moments and center of mass of a region under a curve using approximating rectangles.





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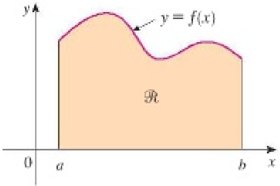

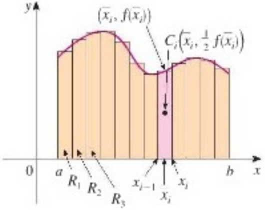

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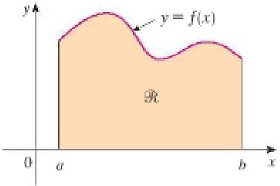

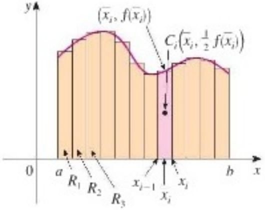

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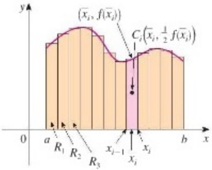

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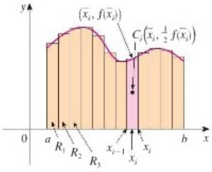

- The rectangle, R_i, above the subinterval $[x_{i-1}, x_i]$ has height $f(\frac{x_{i-1}+x_i}{2}) = f(\bar{x})$, where $\bar{x} = \frac{x_{i-1}+x_i}{2}$. For this rectangle

$$\text{Centroid} = C_i(\bar{x}_i, \frac{1}{2} f(\bar{x}_i)), \quad \text{Area} = \Delta x f(\bar{x}_i), \quad \text{mass} = \rho \Delta x f(\bar{x}_i).$$

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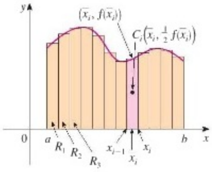

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- $M_y(R_i) = \text{ mass} \times \bar{x}_i = \rho \Delta x f(\bar{x}_i)\bar{x}_i = \rho \bar{x}_i \Delta x f(\bar{x}_i).$
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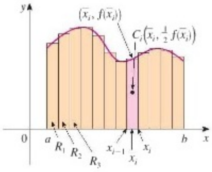
2-D systems regions under curves



- The rectangle, R_i, above the subinterval $[x_{i-1}, x_i]$ has height $f(\frac{x_{i-1}+x_i}{2}) = f(\bar{x})$, where $\bar{x} = \frac{x_{i-1}+x_i}{2}$. For this rectangle

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- $M_y = \lim_{n\to\infty}\sum_{i=1}^n \rho\bar{x}_i f(\bar{x}_i)\Delta x = \rho\int_a^b xf(x)dx$.
  $M_x = \lim_{n\to\infty}\sum_{i=1}^n \frac{\rho}{2}[f(\bar{x}_i)]^2\Delta x = \frac{\rho}{2}\int_a^b [f(x)]^2 dx$.

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For the region beneath a continuous function f which has values greater than or equal to 0 on the interval $[a, b]$, we have

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Example Find the centroid of the region bounded by the curve $y = \frac{1}{x^2}$, the x-axis, the line $x = 1$ and the line $x = 2$.

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- $\bar{x} = \frac{M_y}{A} = \frac{\ln 2}{1/2} = 2\ln 2 \approx 1.3863$. $\quad \bar{y} = \frac{M_x}{A} = \frac{7/48}{1/2} = \frac{7}{24} \approx .29$.

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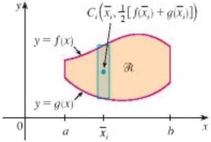
- Centroid of the region $= (\bar{x}, \bar{y}) = (2\ln 2, \frac{7}{24}) \approx (1.3863, 0.29)$.

2-D systems regions between curves



If the region $\mathfrak{R}$ is bounded above by the continuous curve $y = f(x)$ and below by the continuous curve $y = g(x)$, where $f(x) \geq g(x) \geq 0$, we have the moments of a plate with that shape and constant density ρ are given by

$$\boxed{M_y = \rho \int_a^b x[f(x) - g(x)]dx \quad \text{and} \quad M_x = \frac{\rho}{2} \int_a^b [f(x)]^2 - [g(x)]^2 dx.}$$

and the centroid of the region $\mathfrak{R}$ is given by

$$\boxed{\bar{x} = \frac{1}{A} \int_a^b x[f(x) - g(x)]dx \qquad \bar{y} = \frac{1}{2A} \int_a^b [f(x)]^2 - [g(x)]^2 dx}$$

where A denotes the area of the region $\mathfrak{R}$, $\quad A = \int_a^b f(x) - g(x)dx$.

2-D systems regions between curves

Example Find the centroid of the region bounded above by $y = x + 2$ and below by the curve $y = x^2$.

2-D systems regions between curves

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- The curves meet at the points where

$$x + 2 = x^2 \text{ or } x^2 - x - 2 = 0 \text{ or } (x-2)(x+1) = 0 \text{ or } x = -1, \; x = 2.$$

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- The area of the region is given by:

$$A = \int_{-1}^{2} x + 2 - x^2 dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_{-1}^{2} = \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{3}{2} + 6 - \frac{9}{3} = \frac{27}{6}$$

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- Again for convenience we assume that the region has uniform density $\rho = 1$ in our calculations of moments.

$$M_y = \int_{-1}^{2} x[x+2-x^2]dx = \int_{-1}^{2} x^2+2x-x^3 dx = \frac{x^3}{3}+2\frac{x^2}{2}-\frac{x^4}{4}\Big|_{-1}^{2}$$

$$= \frac{8}{3}+4-\frac{16}{4}-\left[\frac{-1}{3}+1-\frac{1}{4}\right] = \frac{9}{3}+3-\frac{15}{4} = \frac{36+36-45}{12} = \frac{27}{12} = 2.25.$$

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Example Find the centroid of the region bounded above by $y = x + 2$ and below by the curve $y = x^2$.

- The curves meet at the points where $x = -1,\ x = 2$.
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- $$M_x = \frac{1}{2}\int_{-1}^{2} [x+2]^2 - [x^2]^2 dx = \frac{1}{2}\int_{-1}^{2} x^2 + 4x + 4 - x^4 dx$$
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$$= \frac{1}{2}\left[\frac{9}{3} + 18 - \frac{33}{5}\right] = \frac{1}{2}\left[21 - \frac{33}{5}\right] = \frac{1}{2}\left(\frac{72}{5}\right) = 7.2.$$

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- $\bar{x} = \frac{M_y}{A} = \frac{27/12}{27/6} = 1/2. \quad \bar{y} = \frac{M_x}{A} = \frac{72/10}{27/6} = 8/5.$

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