Definition and properties of ln(x).

We define a new function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

This function is called the natural logarithm. We derive a number of properties of this new function:

Definition and properties of $\ln(x)$.

We define a new function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

This function is called the natural logarithm. We derive a number of properties of this new function:

- $Domain = (0, \infty)$

Definition and properties of $\ln(x)$.

We define a new function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

This function is called the natural logarithm. We derive a number of properties of this new function:

- $Domain = (0, \infty)$
- $\ln x > 0$ *if* $x > 1$, $\ln x = 0$ *if* $x = 1$, $\ln x < 0$ *if* $x < 1$.

Definition and properties of ln(x).

We define a new function

$$\ln x = \int_1^x \frac{1}{t} dt, \qquad x > 0.$$

This function is called the natural logarithm. We derive a number of properties of this new function:

- $Domain = (0, \infty)$
- $\ln x > 0$ *if* $x > 1$, $\ln x = 0$ *if* $x = 1$, $\ln x < 0$ *if* $x < 1$.
- $\frac{d(\ln x)}{dx} = \frac{1}{x}$

Definition and properties of $\ln(x)$.

We define a new function

$$\ln x = \int_1^x \frac{1}{t} dt, \qquad x > 0.$$

This function is called the natural logarithm. We derive a number of properties of this new function:

- *Domain* $= (0, \infty)$
- $\ln x > 0$ *if* $x > 1$, $\ln x = 0$ *if* $x = 1$, $\ln x < 0$ *if* $x < 1$.
- $\frac{d(\ln x)}{dx} = \frac{1}{x}$
- *The graph of* $y = \ln x$ *is increasing, continuous and concave down on the interval* $(0, \infty)$.

Definition and properties of ln(x).

We define a new function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

This function is called the natural logarithm. We derive a number of properties of this new function:

- *Domain* $= (0, \infty)$
- $\ln x > 0$ *if* $x > 1$, $\ln x = 0$ *if* $x = 1$, $\ln x < 0$ *if* $x < 1$.
- $\frac{d(\ln x)}{dx} = \frac{1}{x}$
- *The graph of* $y = \ln x$ *is increasing, continuous and concave down on the interval* $(0, \infty)$.
- *The function* $f(x) = \ln x$ *is a one-to-one function*
- *Since* $f(x) = \ln x$ *is a one-to-one function, there is a unique number,* e*, with the property that*

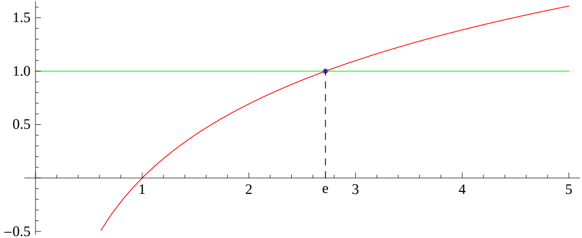
$$\boxed{\ln e = 1.}$$

Graph of $\ln(x)$.

Using the information derived above, we can sketch a graph of the natural logarithm

Graph of $\ln(x)$.

Using the information derived above, we can sketch a graph of the natural logarithm



Rules of Logarithms

We also derived the following algebraic properties of our new function by comparing derivatives. We can use these algebraic rules to simplify the natural logarithm of products and quotients:

Rules of Logarithms

We also derived the following algebraic properties of our new function by comparing derivatives. We can use these algebraic rules to simplify the natural logarithm of products and quotients:

- $\ln 1 = 0$

Rules of Logarithms

We also derived the following algebraic properties of our new function by comparing derivatives. We can use these algebraic rules to simplify the natural logarithm of products and quotients:

- $\ln 1 = 0$
- $\ln(ab) = \ln a + \ln b$

Rules of Logarithms

We also derived the following algebraic properties of our new function by comparing derivatives. We can use these algebraic rules to simplify the natural logarithm of products and quotients:

- $\ln 1 = 0$
- $\ln(ab) = \ln a + \ln b$
- $\ln a^r = r \ln a$

Example using definition of e and rule 3

Example Evaluate $\int_1^{e^2} \frac{1}{t}\,dt$

Example using definition of e and rule 3

Example Evaluate $\int_1^{e^2} \frac{1}{t} dt$

- *From the definition of* $\ln(x)$, *we have*

$$\int_1^{e^2} \frac{1}{t} dt = \ln(e^2)$$

Example using definition of e and rule 3

Example Evaluate $\int_1^{e^2} \frac{1}{t} dt$

- *From the definition of* $\ln(x)$, *we have*

$$\int_1^{e^2} \frac{1}{t} dt = \ln(e^2)$$

-

$$= 2 \ln e = 2.$$

Limits at ∞ and 0.

We can use the rules of logarithms given above to derive the following information about limits.

$$\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0} \ln x = -\infty.$$

(see notes for a proof)
Example Find the limit $\lim_{x\to\infty} \ln(\frac{1}{x^2+1})$.

Limits at ∞ and 0.

We can use the rules of logarithms given above to derive the following information about limits.

$$\boxed{\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0} \ln x = -\infty.}$$

(see notes for a proof)

Example Find the limit $\lim_{x\to\infty} \ln(\frac{1}{x^2+1})$.

- *As* $x \to \infty$, *we have* $\frac{1}{x^2+1} \to 0$

Limits at ∞ and 0.

We can use the rules of logarithms given above to derive the following information about limits.

$$\boxed{\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0} \ln x = -\infty.}$$

(see notes for a proof)
Example Find the limit $\lim_{x\to\infty} \ln(\frac{1}{x^2+1})$.

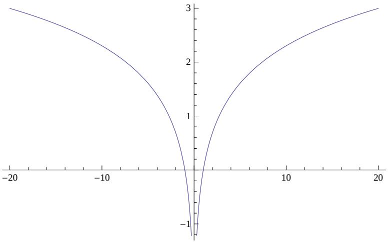
- *As* $x \to \infty$, *we have* $\frac{1}{x^2+1} \to 0$
- *Therfore as* $x \to \infty$, $\ln(\frac{1}{x^2+1}) \to -\infty \quad [= \lim_{u\to 0} \ln(u)]$

$\ln|x|$

We can extend the applications of the natural logarithm function by composing it with the absolute value function. We have :

$$\ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

This is an even function with graph



We have $ln|x|$ is also an antiderivative of $1/x$ with a larger domain than $ln(x)$.

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}} \quad \text{and} \quad \boxed{\int \frac{1}{x}dx = \ln|x| + C}$$

Using Chain Rule

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}} \quad \text{and} \quad \boxed{\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}}$$

Example Differentiate $\ln|\sin x|$.

Example Differentiate $\ln|\sqrt[3]{x-1}|$.

Using Chain Rule

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}} \quad \text{and} \quad \boxed{\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}}$$

Example Differentiate $\ln|\sin x|$.

- *Using the chain rule, we have*

$$\frac{d}{dx}\ln|\sin x| = \frac{1}{\sin x}\frac{d}{dx}\sin x$$

Example Differentiate $\ln|\sqrt[3]{x-1}|$.

Using Chain Rule

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}} \quad \text{and} \quad \boxed{\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}}$$

Example Differentiate $\ln|\sin x|$.

- *Using the chain rule, we have*

$$\frac{d}{dx}\ln|\sin x| = \frac{1}{\sin x}\frac{d}{dx}\sin x$$

- $$= \frac{\cos x}{\sin x}$$

Example Differentiate $\ln|\sqrt[3]{x-1}|$.

Using Chain Rule

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}} \quad \text{and} \quad \boxed{\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}}$$

Example Differentiate $\ln|\sin x|$.

- *Using the chain rule, we have*

$$\frac{d}{dx}\ln|\sin x| = \frac{1}{\sin x}\frac{d}{dx}\sin x$$

- $$= \frac{\cos x}{\sin x}$$

Example Differentiate $\ln|\sqrt[3]{x-1}|$.

- *We can simplify this to finding* $\frac{d}{dx}\left(\frac{1}{3}\ln|x-1|\right)$*, since* $\ln|\sqrt[3]{x-1}| = \ln|x-1|^{1/3}$

Using Chain Rule

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}} \quad \text{and} \quad \boxed{\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}}$$

Example Differentiate $\ln|\sin x|$.

- *Using the chain rule, we have*

$$\frac{d}{dx}\ln|\sin x| = \frac{1}{\sin x}\frac{d}{dx}\sin x$$

-

$$= \frac{\cos x}{\sin x}$$

Example Differentiate $\ln|\sqrt[3]{x-1}|$.

- *We can simplify this to finding* $\frac{d}{dx}\left(\frac{1}{3}\ln|x-1|\right)$*, since* $\ln|\sqrt[3]{x-1}| = \ln|x-1|^{1/3}$
-

$$\frac{d}{dx}\frac{1}{3}\ln|x-1| = \frac{1}{3}\frac{1}{(x-1)}\frac{d}{dx}(x-1) = \frac{1}{3(x-1)}$$

Using Substitution

$$\boxed{\int \frac{1}{x}dx = \ln|x| + C} \quad \text{and} \quad \boxed{\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C}$$

Example Find the integral

$$\int \frac{x}{3 - x^2}dx.$$

Using Substitution

$$\boxed{\int \frac{1}{x} dx = \ln|x| + C} \quad \text{and} \quad \boxed{\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C}$$

Example Find the integral

$$\int \frac{x}{3 - x^2} dx.$$

- *Using substitution, we let $u = 3 - x^2$.*

$$du = -2x\, dx, \qquad x\, dx = \frac{du}{-2},$$

Using Substitution

$$\int \frac{1}{x} dx = \ln|x| + C \quad \text{and} \quad \int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C$$

Example Find the integral

$$\int \frac{x}{3 - x^2} dx.$$

- *Using substitution, we let* $u = 3 - x^2$.

$$du = -2x\, dx, \qquad x\, dx = \frac{du}{-2},$$

- $$\int \frac{x}{3 - x^2} dx = \int \frac{1}{-2(u)}\, du$$

Using Substitution

$$\boxed{\int \frac{1}{x}dx = \ln|x| + C} \quad \text{and} \quad \boxed{\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C}$$

Example Find the integral

$$\int \frac{x}{3 - x^2}dx.$$

- *Using substitution, we let* $u = 3 - x^2$.

$$du = -2x\ dx, \qquad x\ dx = \frac{du}{-2},$$

- $$\int \frac{x}{3 - x^2}dx = \int \frac{1}{-2(u)}\ du$$

- $$= \frac{-1}{2}\ln|u| + C = \frac{-1}{2}\ln|3 - x^2| + C$$

Logarithmic differentiation

To differentiate $y = f(x)$, it is often easier to use logarithmic differentiation :

1. Take the natural logarithm of both sides to get $\ln y = \ln(f(x))$.
2. Differentiate with respect to x to get $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln(f(x))$
3. We get $\frac{dy}{dx} = y\frac{d}{dx}\ln(f(x)) = f(x)\frac{d}{dx}\ln(f(x))$.

Logarithmic differentiation

Example Find the derivative of $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$.

Logarithmic differentiation

Example Find the derivative of $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$.

- *We take the natural logarithm of both sides to get*

$$\ln y = \ln \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

Logarithmic differentiation

Example Find the derivative of $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$.

- *We take the natural logarithm of both sides to get*

$$\ln y = \ln \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

- *Using the rules of logarithms to expand the R.H.S. we get*

$$\ln y = \frac{1}{4}\ln\frac{x^2+1}{x^2-1} = \frac{1}{4}\Big[\ln(x^2+1) - \ln(x^2-1)\Big] = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1)$$

Logarithmic differentiation

Example Find the derivative of $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$.

- *We take the natural logarithm of both sides to get*

$$\ln y = \ln \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

- *Using the rules of logarithms to expand the R.H.S. we get*

$$\ln y = \frac{1}{4} \ln \frac{x^2+1}{x^2-1} = \frac{1}{4}\left[\ln(x^2+1) - \ln(x^2-1)\right] = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1)$$

- *Differentiating both sides with respect to x, we get*

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4}\cdot\frac{2x}{(x^2+1)} - \frac{1}{4}\cdot\frac{2x}{(x^2-1)} = \frac{x}{2(x^2+1)} - \frac{x}{2(x^2-1)}$$

Logarithmic differentiation

Example Find the derivative of $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$.

- *We take the natural logarithm of both sides to get*

$$\ln y = \ln \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

- *Using the rules of logarithms to expand the R.H.S. we get*

$$\ln y = \frac{1}{4} \ln \frac{x^2+1}{x^2-1} = \frac{1}{4}\Big[\ln(x^2+1) - \ln(x^2-1)\Big] = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1)$$

- *Differentiating both sides with respect to x, we get*

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4}\cdot\frac{2x}{(x^2+1)} - \frac{1}{4}\cdot\frac{2x}{(x^2-1)} = \frac{x}{2(x^2+1)} - \frac{x}{2(x^2-1)}$$

- *Multiplying both sides by y, we get*

$$\frac{dy}{dx} = y\left[\frac{x}{2(x^2+1)} - \frac{x}{2(x^2-1)}\right]$$

Logarithmic differentiation

Example Find the derivative of $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$.

- *We take the natural logarithm of both sides to get*

$$\ln y = \ln \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

- *Using the rules of logarithms to expand the R.H.S. we get*

$$\ln y = \frac{1}{4} \ln \frac{x^2+1}{x^2-1} = \frac{1}{4}\Big[\ln(x^2+1) - \ln(x^2-1)\Big] = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1)$$

- *Differentiating both sides with respect to x, we get*

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4} \cdot \frac{2x}{(x^2+1)} - \frac{1}{4} \cdot \frac{2x}{(x^2-1)} = \frac{x}{2(x^2+1)} - \frac{x}{2(x^2-1)}$$

- *Multiplying both sides by y, we get*

$$\frac{dy}{dx} = y\left[\frac{x}{2(x^2+1)} - \frac{x}{2(x^2-1)}\right]$$

- *Converting y to a function of x, we get*

$$\frac{dy}{dx} = \sqrt[4]{\frac{x^2+1}{x^2-1}}\left[\frac{x}{2(x^2+1)} - \frac{x}{2(x^2-1)}\right]$$

Summary of formulas

$$\boxed{\ln(x)}$$

$$\ln(ab) = \ln a + \ln b, \quad \ln(\frac{a}{b}) = \ln a - \ln b$$

$$\ln a^x = x \ln a$$

$$\lim_{x \to \infty} \ln x = \infty, \quad \lim_{x \to 0} \ln x = -\infty$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad \frac{d}{dx} \ln |g(x)| = \frac{g'(x)}{g(x)}$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int \frac{g'(x)}{g(x)} dx = \ln |g(x)| + C.$$

Summary of methods

Logarithmic Differentiation
(Finding formulas for inverse functions)
Finding slopes of inverse functions (using formula from lecture 1).
Calculating Limits
Calculating Derivatives
Calculating Integrals (including definite integrals)