First Order Linear Differential Equations

A **First Order Linear Differential Equation** is a first order differential equation which can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x), Q(x)$ are continuous functions of x on a given interval.
The above form of the equation is called the **Standard Form** of the equation.
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- Integrating both sides with respect to x, we get $\int \frac{d[I(x)y]}{dx} dx = \int I(x)Q(x)dx$ or $I(x)y = \int I(x)Q(x)dx + C$ giving us a solution of the form

$$\boxed{y = \frac{\int I(x)Q(x)dx + C}{I(x)}}$$

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- Hence our solution is

$$y = -x^2 + Cx^3$$

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- $y(0) = -6 \quad \rightarrow \quad 1 + C = -6 \quad \rightarrow \quad C = -7 \quad \rightarrow \quad \boxed{y = 1 - 7e^{-x^2/2}}$.

Old Exam Question: Q 13 Exam 2 Spring 2008

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- Multiply the standard equation by $I(x) = \pm(x+1)$ to get $(x+1)\frac{dy}{dx} + y = 2x \qquad \text{or} \qquad \frac{d(x+1)y}{dx} = 2x.$

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- $y(1) = 2 \quad \to \quad \frac{1+C}{2} = 2 \quad \to \quad C = 3 \quad \to \quad \boxed{y = \frac{x^2+3}{x+1}}$.