First Order Linear Differential Equations

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$$\frac{dy}{dx} + P(x)y = Q(x)$$

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- Integrating both sides with respect to x, we get $\int \frac{d[I(x)y]}{dx}dx = \int I(x)Q(x)dx$ or $I(x)y = \int I(x)Q(x)dx + C$ giving us a solution of the form

$$\boxed{y = \frac{\int I(x)Q(x)dx + C}{I(x)}}$$

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- Hence our solution is

$$y = -x^2 + Cx^3$$

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- We get $e^{x^2/2}y = e^{x^2/2} + C \quad \rightarrow y = 1 + Ce^{-x^2/2}$.
- $y(0) = -6 \quad \rightarrow \quad 1 + C = -6 \quad \rightarrow \quad C = -7 \quad \rightarrow \quad \boxed{y = 1 - 7e^{-x^2/2}}$.

Second Order Linear Differential Equations

(Section 17.1), see e-book

A **Second Order Linear Differential Equation** is a second order differential equation which can be put in the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

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- If $G = 0$ then the equation is called **homogeneous**. Otherwise is called **nonhomogeneous**.
- In this lecture, we will solve homogeneous second order linear equations, in the next lecture, we will cover nonhomogeneous second order linear equations.
- In general is very difficult to solve second order linear equations, general ones will be solved in a differential equations class. In this course we restrict attention to second order linear equations with constant coefficients, today we study equations of the type:

$$ay'' + by' + cy = 0,$$

where a, b and c are constants and $a \neq 0$.

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- **Theorem** If $y_1(x)$ and $y_2(x)$ are solutions to the differential equation $ay'' + by' + cy = 0$, then every function of the form $c_1y_1(x) + c_2y_2(x)$, where c_1, c_2 are constants, is also a solution of the equation.

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- **Proof**
  $ay'' + by' + cy = c_1[ay_1'' + by_1' + cy_1] + c_2[ay_2'' + by_2' + cy_2] = 0 + 0 = 0.$

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- **Proof**
  $ay'' + by' + cy = c_1[ay_1'' + by_1' + cy_1] + c_2[ay_2'' + by_2' + cy_2] = 0 + 0 = 0.$
- **Theorem** Let $y_1(x)$ and $y_2(x)$ are (non-zero) solutions to the differential equation $ay'' + by' + cy = 0$, where $y_1 \neq cy_2$ (equivalently $y_2 \neq ky_1$), for constants c and k. Then the general solution to the differential equation $ay'' + by' + cy = 0$ is given by

$$c_1y_1(x) + c_2y_2(x).$$

Note: If $y_1 \neq cy_2$ for any real number $c \neq 0$, we say y_1 and y_2 are **linearly independent** solutions.

Second Order Linear Differential Equations

Example: Harmonic motion Check that $y_1(x) = \sin(\sqrt{2}x)$ and $y_2(x) = \cos(\sqrt{2}x)$ are solutions to the differential equation

$$\frac{d^2y}{dx^2} + 2y = 0.$$

Since $\sin(\sqrt{2}x) \neq C\cos(\sqrt{2}x)$, we have the general solution to this differential equation is given by

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- $y_1'(x) = \sqrt{2}\cos(\sqrt{2}x), \quad y_1''(x) = -2\sin(\sqrt{2}x) = -2y_1(x)$. Therefore $y_1(x)$ is a solution to the above equation. Similarly $y_2(x)$ is a solution to the equation.

Auxiliary Equation/Characteristic equation

To solve a second order differential equation of the form

$$ay'' + by' + cy = 0, \qquad a \neq 0,$$

we must consider the roots of **auxiliary equation (or characteristic equation)** given by $ar^2 + br + c = 0$.

Example What are the roots of the auxiliary equation of the differential equation $y'' + 2y' - 8y = 0$?

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$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

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- We demonstrate how to extract 2 linearly independent solutions from the auxiliary equation in each of 3 possible cases below. The results depend on the sign of the **discriminant** $b^2 - 4ac$.

Auxiliary Equation/Characteristic equation

Method for solving : Given a homogeneous second order linear equation with constant coefficients

$$ay'' + by + c = 0, \quad a \neq 0$$

Step 1: Write down the auxiliary equation $ar^2 + br + c = 0$. Calculate the discriminant $b^2 - 4ac$.

Step 2: Calculate r_1 and r_2 above.

Step 3: If $b^2 - 4ac > 0$, ($r_1 \neq r_2$, both real) In this case, $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are linearly independent solutions. General solution:

$$c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

If $b^2 - 4ac = 0$, ($r = r_1 = r_2$, real). In this case $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions (check book for details). General Solution:

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If $b^2 - 4ac < 0$, (Complex roots, $r_1 = \alpha + i\beta, r_2 = \alpha + i\beta, \quad \alpha, \beta \in \mathbb{R}$, $i = \sqrt{-1}$.). By definition, $e^{\alpha + i\beta} = e^{\alpha}(cos\beta + i \sin \beta)$. General solution $c_1 e^{r_1 x} + c_2 e^{r_2 x}$ can be rearranged (see book for details) to show that all solutions are of the form :

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad c_1, c_2 \text{real or complex}$$

Example 1

Solutions of $ay'' + by' + cy = 0$**:**

Roots of $ar^2 + br + c = 0$	General Solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex : $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

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- Auxiliary equation: $r^2 + 2r - 8 = 0$, roots $(r-2)(r+4) = 0$. $r_1 = 2$, $r_2 = -4$.
- The roots are real and distinct, general solution is given by

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{2x} + c_2 e^{-4x}.$$

Example 2

Solutions of $ay'' + by' + cy = 0$**:**

Roots of $ar^2 + br + c = 0$	General Solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex : $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

Example Solve the differential equation $y'' - 6y' + 9y = 0$.

Example 2

Solutions of $ay'' + by' + cy = 0$**:**

Roots of $ar^2 + br + c = 0$	General Solution
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$$y = c_1 e^{rx} + c_2 x e^{rx} = c_1 e^{3x} + c_2 x e^{3x}.$$

Example 3

Solutions of $ay'' + by' + cy = 0$**:**

Roots of $ar^2 + br + c = 0$	General Solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
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r_1, r_2 complex : $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

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- Auxiliary equation: $r^2 + 3 = 0$, roots: $r = \pm\sqrt{-3} = \pm i\sqrt{3}$. Complex roots, $\alpha = 0$, $\beta = \sqrt{3}$.

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- Auxiliary equation: $r^2 + 3 = 0$, roots: $r = \pm\sqrt{-3} = \pm i\sqrt{3}$. Complex roots, $\alpha = 0$, $\beta = \sqrt{3}$.
- General solution:

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) = e^{0x}(c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x))$$

$$= c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x).$$

Example 4

Solutions of $ay'' + by' + cy = 0$:

Roots of $ar^2 + br + c = 0$	General Solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
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r_1, r_2 complex : $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

Example Solve the differential equation $y'' + y' + 3y = 0$.

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Example 4

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Example Solve the differential equation $y'' + y' + 3y = 0$.

- Auxiliary equation: $r^2 + r + 3 = 0$.
- Roots:

$$\frac{-1 \pm \sqrt{1 - 4(1)(3)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-11}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{11}}{2}.$$

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- General solution:

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{11}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{11}}{2}x\right)\right].$$

Initial value problems

An Initial Value Problem for a second-order differential equations asks for a specific solution to the differential equation that also satisfies **TWO initial conditions** of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

Note: for a differential equation of type $ay'' + by' + cy = G(x), \quad a \neq 0$, a solution to an initial value problem always exists and is unique. You will see a proof of this in later courses on differential equations.

Initial value problems, Example

Example Solve the initial value problem
$2y'' + 3y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2.$

Initial value problems, Example

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- Auxiliary equation: $2r^2 + 3r + 1 = 0$. Roots: $\frac{-3\pm\sqrt{9-8}}{4} = \frac{-3\pm 1}{4} = \{-1, \ -1/2\}$. General Solution: $\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{-x} + c_2 e^{-\frac{1}{2}x}}$.

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- (InVal)$y(0) = 1 \to c_2 e^0 + c_2 e^0 = 1 \to \boxed{c_1 + c_2 = 1.}$

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- (InVal)$y(0) = 1 \to c_2 e^0 + c_2 e^0 = 1 \to \boxed{c_1 + c_2 = 1.}$
- $y'(x) = -c_1 e^{-x} - \frac{1}{2} c_2 e^{-\frac{1}{2}x}$, (InVal)

  $$y'(0) = 2 \quad \to \quad -c_1 e^0 - \tfrac{1}{2} c_2 e^0 = 2 \quad \to \quad \boxed{-c_1 - \frac{1}{2} c_2 = 2.}$$

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- Adding the equations, I get $\frac{1}{2} c_2 = 3 \rightarrow \boxed{c_2 = 6.}$

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- Substituting this into the first equation, I get $c_1 + 6 = 1 \quad \to \quad \boxed{c_1 = -5.}$
- My solution to this initial value problem is

$$\boxed{y = -5e^{-x} + 6e^{-\frac{1}{2}x}.}$$

Boundary Value Problems

A **boundary value problem** for a second-order differential equations asks for a specific solution to the differential equation that also satisfies **TWO conditions** of the form

$$y(x_0) = y_0, \quad y(x_1) = y_1.$$

Note Such a boundary value problem does not always have a solution.

Boundary Value Problems, Example

Example Find a solution to the boundary value problem,

$$y'' + 3y = 0, \quad y(0) = 3, \quad y\left(\frac{\pi}{2\sqrt{3}}\right) = 5.$$

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- Using the boundary values, we get :

$$y(0) = 3 \;\rightarrow\; c_1 \cos(0) + c_2 \sin(0) = 3 \quad \rightarrow \quad \boxed{c_1 = 3.}$$

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- $y\left(\frac{\pi}{2\sqrt{3}}\right) = 5 \;\rightarrow\; c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = 5 \;\rightarrow\; \boxed{c_2 = 5.}$

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- The solution is given by

$$y = 3\cos(\sqrt{3}x) + 5\sin(\sqrt{3}x).$$