NonHomogeneous Linear Equations (Section 17.2)

The solution of a second order **nonhomogeneous** linear differential equation of the form

$$ay'' + by' + cy = G(x)$$

where a, b, c are constants, $a \neq 0$ and $G(x)$ is a continuous function of x on a given interval is of the form

$$y(x) = y_p(x) + y_c(x)$$

where $y_p(x)$ is a particular solution of $ay'' + by' + cy = G(x)$ and $y_c(x)$ is the general solution of the **complementary equation/ corresponding homogeneous equation** $ay'' + by' + cy = 0$.

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- Since we already know how to find y_c, the general solution to the corresponding homogeneous equation, we need a method to find a particular solution, y_p, to the equation. One such methods is described below. This method may not always work. A second method which is always applicable is demonstrated in the extra examples in your notes.

The method of Undetermined Coefficients

We wish to search for a particular solution to $ay'' + by' + cy = G(x)$.
If $G(x)$ is a polynomial it is reasonable to guess that there is a particular solution, $y_p(x)$ which is a polynomial in x of the same degree as $G(x)$ (because if y is such a polynomial, then $ay'' + by' + c$ is also a polynomial of the same degree.)
Method to find a particular solution: Substitute $y_p(x) =$ a polynomial of the same degree as G into the differential equation and determine the coefficients.
Example Solve the differential equation: $y'' + 3y' + 2y = x^2$.

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Method to find a particular solution: Substitute $y_p(x) =$ a polynomial of the same degree as G into the differential equation and determine the coefficients.
Example Solve the differential equation: $y'' + 3y' + 2y = x^2$.

- We first find the solution of the complementary/ corresponding homogeneous equation, $y'' + 3y' + 2y = 0$:
  Auxiliary equation: $r^2 + 3r + 2 = 0$
  Roots: $(r + 1)(r + 2) = 0 \quad \rightarrow \quad r_1 = -1, \quad r_2 = -2$. Distinct real roots.
  Solution to corresponding homogeneous equation:
  $\boxed{y_c = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{-x} + c_2 e^{-2x}}$.

Undetermined coefficients Example (polynomial)

$$y(x) = y_p(x) + y_c(x)$$

Example Solve the differential equation: $y'' + 3y' + 2y = x^2$.

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{-x} + c_2 e^{-2x}$$

We now need a particular solution $y_p(x)$.

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- We consider a trial solution of the form $y_p(x) = Ax^2 + Bx + C$.

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- Then $y_p'(x) = 2Ax + B, \quad y_p''(x) = 2A$.

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- Then $y_p'(x) = 2Ax + B, \quad y_p''(x) = 2A$.
- We plug y_p'', y_p' and y_p into the equation to get $2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$

$$\rightarrow \quad 2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2.$$

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  $2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$
  $$\rightarrow \quad 2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2.$$
- Equating coefficients, we get
  $$2A = 1 \rightarrow \boxed{A = \frac{1}{2}}, \quad 6A + 2B = 0, \quad 2A + 3B + 2C = 0.$$

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- Equating coefficients, we get
  $$2A = 1 \rightarrow \boxed{A = \frac{1}{2}}, \quad 6A + 2B = 0, \quad 2A + 3B + 2C = 0.$$
- Plugging the value for A into the middle equation, we get
  $$3 + 2B = 0 \quad \rightarrow \quad \boxed{B = -\frac{3}{2}}.$$

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Example Solve the differential equation: $y'' + 3y' + 2y = x^2$.

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- Let us assume $y_p(x) = Ax^2 + Bx + C$.
- Then $y_p'(x) = 2Ax + B$, $y_p''(x) = 2A$.
- We plug y_p'', y_p' and y_p into the equation to get $2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$

$$\rightarrow \quad 2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2.$$

- Equating coefficients, we get $2A = 1 \rightarrow \boxed{A = \frac{1}{2}}$, $6A + 2B = 0$, $2A + 3B + 2C = 0$.
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- Equating coefficients, we get $2A = 1 \rightarrow \boxed{A = \frac{1}{2}}$, $6A + 2B = 0$, $2A + 3B + 2C = 0$.
- Plugging the value for A into the middle equation, we get $3 + 2B = 0 \rightarrow \boxed{B = -\frac{3}{2}}$.
- Using values for A and B in the third equation, we get

$1 - \frac{9}{2} + 2C = 0 \quad \rightarrow \boxed{C = \frac{7}{4}}$.

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- $\boxed{A = \frac{1}{2}}$, $\boxed{B = -\frac{3}{2}}$, $\boxed{C = \frac{7}{4}}$

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- $\boxed{A = \frac{1}{2}}$, $\boxed{B = -\frac{3}{2}}$, $\boxed{C = \frac{7}{4}}$
- Hence a particular solution is given by $\boxed{y_p = \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]}$,

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- $\boxed{A = \frac{1}{2}}, \quad \boxed{B = -\frac{3}{2}}, \quad \boxed{C = \frac{7}{4}}$
- Hence a particular solution is given by $\boxed{y_p = \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]}$,
- and the general solution is given by

$$\boxed{y = y_p + y_c = \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right] + c_1 e^{-x} + c_2 e^{-2x}.}$$

$G(x) = Ce^{kx}$. Example (Exponential)

We wish to search for a particular solution to $ay'' + by' + cy = G(x)$.
If $G(x)$ is of the form Ce^{kx}, where C and k are constants, then we use a trial solution of the form $y_p(x) = Ae^{kx}$ and solve for A if possible.
Example Solve $y'' + 9y = e^{-4x}$.

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Example Solve $y'' + 9y = e^{-4x}$.

- We first find the solution of the corresponding homogeneous equation, $y'' + 9y' = 0$:
  Auxiliary equation: $r^2 + 9 = 0$
  Roots: $r^2 = -9 \quad \rightarrow \quad r_1 = 3i, \quad r_2 = -3i$. Complex roots.
  Solution to corresponding homogeneous equation :
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- We plug y_p'', y_p' and y_p into the equation to get

$$16Ae^{-4x} + 9Ae^{-4x} = e^{-4x} \quad \rightarrow \quad e^{-4x}[16A + 9A] = e^{-4x}.$$

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$$16Ae^{-4x} + 9Ae^{-4x} = e^{-4x} \quad \rightarrow \quad e^{-4x}[16A + 9A] = e^{-4x}.$$

- Since $e^{-4x} > 0$, we can cancel to get $25A = 1$ $\boxed{A = \frac{1}{25}}$.

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- Hence a particular solution is given by : $\boxed{y_p = \frac{1}{25}e^{-4x}}$,
- General solution: $\boxed{y = y_p + y_c = \frac{1}{25}e^{-4x} + c_1 \cos(3x) + c_2 \sin(3x)}$

$G(x) = C\cos kx$ or $C\sin kx$. Example (Trigonometric)

If $G(x)$ is of the form $C\cos kx$ or $C\sin kx$, where C and k are constants, then we use a trial solution of the form $y_p(x) = A\cos(kx) + B\sin(kx)$ and solve for A and B if possible.

Below we use the fact that if $K_1\cos(\alpha x) + K_2\sin(\alpha x) = 0$ for constants K_1, K_2, α, where $\alpha \neq 0$, Then we must have $K_1 = K_2 = 0$.

Example Solve $y'' - 4y' - 5y = \cos(2x)$.

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- We first find the solution of the corresponding homogeneous equation, $y'' - 4y' - 5y = 0$:
  Auxiliary equation: $r^2 - 4r - 5 = 0$
  Roots: $(r+1)(r-5) = 0 \quad \to \quad r_1 = -1, \quad r_2 = 5$. Distinct real roots.
  Solution to corresponding homogeneous equation : $\boxed{y_c = c_1 e^{-x} + c_2 e^{5x}}$.

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- We plug y_p'', y_p' and y_p into the equation to get $-4A\cos(2x) - 4B\sin(2x) - 4[-2A\sin(2x) + 2B\cos(2x)] - 5[A\cos(2x) + B\sin(2x)] = \cos(2x)$

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If $G(x)$ is of the form $C\cos kx$ or $C\sin kx$, where C and k are constants, then we use a trial solution of the form $y_p(x) = A\cos(kx) + B\sin(kx)$ and solve for A and B if possible.
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Example Solve $y'' - 4y' - 5y = \cos(2x)$.

- We first find the solution of the corresponding homogeneous equation, $y'' - 4y' - 5y = 0$:
  Auxiliary equation: $r^2 - 4r - 5 = 0$
  Roots: $(r+1)(r-5) = 0 \quad \rightarrow \quad r_1 = -1, \quad r_2 = 5$. Distinct real roots.
  Solution to corresponding homogeneous equation : $\boxed{y_c = c_1e^{-x} + c_2e^{5x}}$.
- Next we find a particular solution. Our trial solution takes the form: $y_p(x) = A\cos(2x) + B\sin(2x)$.
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- Tidying up, we get
  $(-4A - 8B - 5A - 1)\cos(2x) + (-4B + 8A - 5B)\sin(2x) = 0 \quad \rightarrow$
  $(-9A - 8B - 1)\cos(2x) + (-9B + 8A)\sin(2x) = 0$.

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- We must have $-9A - 8B = 1, \qquad 8A - 9B = 0.$
- From the second equation, we get $B = \frac{8}{9}A$ and substituting this into the first equation, we get

$$-9A - \frac{64}{9}A = 1 \quad \rightarrow \quad -\frac{145}{9}A = 1 \quad \rightarrow \quad \boxed{A = -\frac{9}{145}}.$$

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- Substituting we get $B = \frac{8}{9}A \quad \rightarrow \quad \boxed{B = -\frac{8}{145}}$.
- Hence a particular solution is given by $y_p = -\frac{9}{145}\cos(2x) - \frac{8}{145}\sin(2x)$, and the general solution is given by

  $$\boxed{y = y_p + y_c = -\frac{9}{145}\cos(2x) - \frac{8}{145}\sin(2x) + c_1e^{-x} + c_2e^{5x}.}$$

Trouble: Trial solution fits corresponding hom. eqn.

Troubleshooting

If the trial solution y_p is a solution of the corresponding homogeneous equation, then it cannot be a solution to the non-homogeneous equation. In this case, we multiply the trial solution by x (or x^2 or x^3 ... as necessary) to get a new trial solution that does not satisfy the corresponding homogeneous equation. Then proceed as above.

Example Solve $y'' - y = e^x$.

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- We plug y_p'' and y_p into the equation to get

$$2Ae^x + Axe^x - Axe^x = e^x \quad \to \quad 2Ae^x = e^x \quad \to \quad 2A = 1 \quad \to \quad \boxed{A = 1/2}$$

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- Hence a particular solution is given by $\boxed{y_p = \frac{1}{2}xe^x}$

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- Particular solution: $\boxed{y_p = \frac{1}{2}xe^x}$
- general solution: $\boxed{y = y_p + y_c = \frac{1}{2}xe^x + c_1 e^{-x} + c_2 e^x}$

$G(x) = G_1(x) + G_2(x)$.

To solve the equation $ay'' + by' + cy = G_1(x) + G_2(x)$, we can find particular solutions, y_{p1} and y_{p2} to of the equations

$$ay'' + by' + cy = G_1(x), \qquad ay'' + by' + cy = G_2(x)$$

separately. The general solution of the equation $ay'' + by' + cy = G_1(x) + G_2(x)$ is $y_{p1} + y_{p2} + y_c$, where y_c is the solution of the corresponding homogeneous equation $ay'' + by' + cy = 0$.

Example For the equation $y'' + y' + y = x^2 + e^x$, we use a trial solution of the form

$$y_p = (Ax^2 + Bx + C) + De^x.$$