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In this section, we see that we can sometimes decide whether a series converges or diverges by comparing it to an improper integral. The analysis in this section only applies to **series $\sum a_n$, with positive terms, that is $a_n > 0$.**

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$$f(n) = a_n.$$

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ converges, that is:

$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is convergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is convergent.}$$

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- **Note** The result is still true if the condition that $f(x)$ is decreasing on the interval $[1, \infty)$ is relaxed to "the function $f(x)$ is decreasing on an interval $[M, \infty)$ for some number $M \geq 1$."

Integral Test (Why it works: convergence)

We know from a previous lecture that
$\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \leq 1$.

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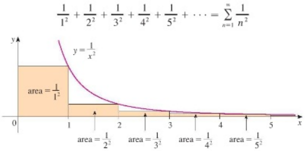

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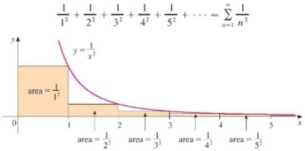

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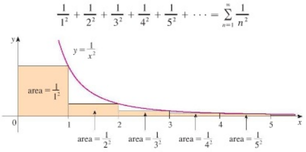

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- Since the sequence $\{s_n\}$ is increasing (because each $a_n > 0$) and bounded, we can conclude that the sequence of partial sums converges and hence the series

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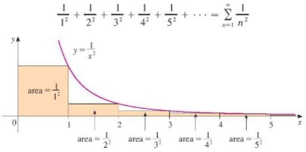

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- **NOTE** We are not saying that $\sum_{i=1}^\infty \frac{1}{n^2} = \int_1^\infty \frac{1}{x^2}\,dx$ here.

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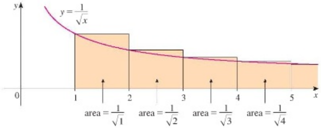

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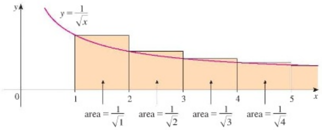

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$$s_n > s_{n-1} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} > \int_1^n \frac{1}{\sqrt{x}}dx$$

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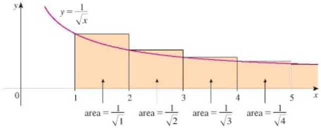

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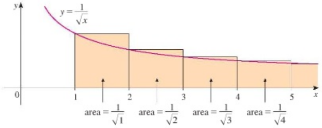

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- Thus we see that $\lim_{n\to\infty} s_n > \lim_{n\to\infty} \int_1^n \frac{1}{\sqrt{x}}dx$.
- However, we know that $\int_1^n \frac{1}{\sqrt{x}}dx$ grows without bound and hence since $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges, we can conclude that $\sum_{k=1}^\infty \frac{1}{\sqrt{n}}$ also diverges.

Integral test, Example.

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- Therefore, the series $\sum_{n=1}^{\infty} \frac{2}{3n+5}$ diverges.

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- Therefore, the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

p-series

We know that $\int_1^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges if $p \leq 1$.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for} \quad p > 1, \quad \text{diverges for} \quad p \leq 1.}$$

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$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{15}}, \qquad \sum_{n=10}^{\infty} \frac{1}{n^{15}}, \qquad \sum_{n=100}^{\infty} \frac{1}{\sqrt[5]{n}},$$

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- $\sum_{n=100}^{\infty} \frac{1}{\sqrt[5]{n}}$ conv/diverges if and only if $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ conv/div. This diverges since $p = 1/5 < 1$.