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$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for} \quad p > 1, \quad \text{diverges for} \quad p \leq 1.$$

$$\sum_{n=1}^{\infty} ar^{n-1} \quad \text{converges if} \quad |r| < 1, \quad \text{diverges if} \quad |r| \geq 1.$$

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- **Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms.**

  (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, than $\sum a_n$ is also convergent.

  (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

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**Limit Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms**. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c$ is a **finite number** and $c > 0$, then either both series converge or both diverge. (Note $c \neq 0$ or $\infty$. )

**Example** Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

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- Since $c = 1 > 0$, we can conclude that both series converge.

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- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} (\frac{n^2+2n+1}{n^4+n^2+2n+1})/(1/n^2) = \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{n^4+n^2+2n+1} = \lim_{n\to\infty} \frac{1+2/n+1/n^2}{1+1/n^2+2/n^3+1/n^4} = 1$.
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- Since $c = 2 > 0$, we can conclude that both series diverge.

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