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- **Example**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots$$

$$\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{2n+1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \dots$$

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- We can use the divergence test to show that the second series above diverges, since

$$\lim_{n \to \infty} (-1)^{n+1} \frac{n}{2n+1} \quad \text{does not exist}$$

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We have the following test for such alternating series:

**Alternating Series test** If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots \qquad b_n > 0$$

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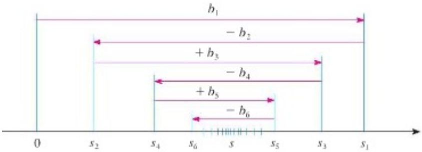
- we see from the graph that because the values of $b_n$ are decreasing, the partial sums of the series cluster about some point in the interval $[0, b_1]$.



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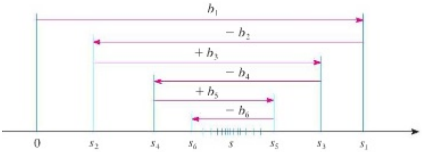


- A proof is given at the end of the notes.

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- Also we really only need $b_{n+1} \leq b_n$ for all $n > N$ for some $N$, since a finite number of terms do not change whether a series converges or not.
- Recall that if we have a differentiable function $f(x)$, with $f(n) = b_n$, then we can use its derivative to check if terms are decreasing.

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**Alternating Series test** If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$ $b_n > 0$ satisfies

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- Note that an alternating series may converge whilst the sum of the absolute values diverges. In particular the alternating harmonic series above converges.

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- Therefore, we can conclude that the alternating series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ converges.

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- Therefore $\lim_{n\to\infty}(-1)^n \frac{2n^2}{n^2+1}$ does not exist and we can conclude that the series
$$\sum_{n=1}^{\infty}(-1)^n \frac{2n^2}{n^2+1}$$
diverges.

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- $b_{n+1} = \frac{1}{(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 1} = \frac{1}{(n+1)} \cdot \frac{1}{n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 1} = \frac{1}{n+1} \cdot b_n < b_n$ if $n > 1$.

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- $b_{n+1} = \frac{1}{(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 1} = \frac{1}{(n+1)} \cdot \frac{1}{n\cdot(n-1)\cdot(n-2)\cdot\dots\cdot 1} = \frac{1}{n+1} \cdot b_n < b_n$ if $n > 1$.
- Therefore by the Alternating series test, we can conclude that the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$ converges.

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- We have $b_n = \frac{\ln n}{n^2}$.
- $\lim_{n\to\infty} \frac{\ln n}{n^2} = \lim_{x\to\infty} \frac{\ln x}{x^2} = (L'Hop) \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$
- To check if $b_n$ is decreasing as $n$ increases, we calculate the derivative of $f(x) = \frac{\ln x}{x^2}$.

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- To check if $b_n$ is decreasing as $n$ increases, we calculate the derivative of $f(x) = \frac{\ln x}{x^2}$.
- $f'(x) = \frac{(x^2)(1/x) - 2x\ln x}{x^2} = \frac{x - 2x\ln x}{x^2} = \frac{x(1-2\ln x)}{x^2} < 0$ if $1 - 2\ln x < 0$ or $\ln x > 1/2$. This happens if $x > \sqrt{e}$, which certainly happens if $x \geq 2$.

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- This is enough to show that $b_{n+1} < b_n$ if $n \geq 2$ and hence $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n^2}$ converges.

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- Therefore $\lim_{n\to\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series $\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{\pi}{n}\right)$ diverges by the divergence test.

# Error of Estimation

### Estimating the Error

Suppose $\sum_{i=1}^{\infty}(-1)^{n-1}b_n,\ b_n > 0$, converges to $s$. Recall that we can use the partial sum $s_n = b_1 - b_2 + \cdots + (-1)^{n-1}b_n$ to estimate the sum of the series, $s$. If the series satisfies the conditions for the Alternating series test, we have the following simple estimate of the size of **the error in our approximation** $|R_n| = |s - s_n|$.
($R_n$ here stands for the remainder when we subtract the $n$ th partial sum from the sum of the series. )

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**Alternating Series Estimation Theorem** If $s = \sum(-1)^{n-1}b_n, \quad b_n > 0$ is the sum of an alternating series that satisfies

$$(i) \quad b_{n+1} < b_n \quad \text{for all } n$$

$$(ii) \quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

A proof is included at the end of the notes.

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- Therefore, if we find a value of $n$ for which $\frac{1}{n+1} \le \frac{1}{10^2}$, we will have the error of approximation $R_n \le \frac{1}{10^2}$.

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- Therefore, if we find a value of $n$ for which $\frac{1}{n+1} \leq \frac{1}{10^2}$, we will have the error of approximation $R_n \leq \frac{1}{10^2}$.
- $\frac{1}{n+1} \leq \frac{1}{10^2}$ if $10^2 \leq n + 1$ or $n \geq 101$.

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- Therefore, if we find a value of $n$ for which $\frac{1}{n+1} \leq \frac{1}{10^2}$, we will have the error of approximation $R_n \leq \frac{1}{10^2}$.
- $\frac{1}{n+1} \leq \frac{1}{10^2}$ if $10^2 \leq n + 1$ or $n \geq 101$.
- Checking with Mathematica, we get the actual error $R_{101} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} - \sum_{n=1}^{101} (-1)^n \frac{1}{n} = 0.00492599$ which is indeed less than .01.