Absolute convergence

Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
If the terms of the series a_n are positive, absolute convergence is the same as convergence.

Example Are the following series absolutely convergent?

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- To check if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent, we need to check if the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

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- Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3 > 1$, it converges and therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent.

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- To check if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is absolutely convergent, we need to check if the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent.
- Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$, it diverges and therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is not absolutely convergent.

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Which of the series in the above example is conditionally convergent?

- Since the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is absolutely convergent, it is **not conditionally convergent.**
- Since the series $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n}$ is convergent (used the alternating series test last day to show this), but the series of absolute values $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n}$ **is conditionally convergent.**

Absolute conv. implies conv.

TheoremIf a series is absolutely convergent, then it is convergent, that is if $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.
(A proof is given in your notes)
Example Are the following series convergent (test for absolute convergence)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{\sin(n)}{n^4}.$$

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- Therefore the series $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^4}\right|$ converges by comparison with the converging p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

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- Therefore the series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^4}$ is convergent since it is absolutely convergent.

The Ratio Test

This test is useful for determining absolute convergence.
Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).
Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

- If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
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- If $L = 1$, then the Ratio test is inconclusive and we cannot determine if the series converges or diverges using this test.

This test is especially useful where factorials and powers of a constant appear in terms of a series. (Note that when the ratio test is inconclusive for an alternating series, the alternating series test may work.)

Example 1 Test the following series for convergence

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- $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x} \ln(1 + 1/x) = e^{\lim_{x\to\infty} x \ln(1+1/x)}.$

  $\lim_{x\to\infty} x \ln(1 + 1/x) = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} = (L'Hop) \lim_{x\to\infty} \frac{\frac{-1/x^2}{(1+1/x)}}{-1/x^2} =$ $\lim_{x\to\infty} \frac{1}{(1+1/x)} = 1.$

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  $\lim_{x\to\infty} x \ln(1 + 1/x) = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} = (L'Hop) \lim_{x\to\infty} \frac{\frac{-1/x^2}{(1+1/x)}}{-1/x^2} =$ $\lim_{x\to\infty} \frac{1}{(1+1/x)} = 1$.
- Therefore $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e^1 = e > 1$ and the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

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- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)^2}{1/n^2}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^2 = 1.$

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- Lets see what happens when we apply the ratio test here.
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1\big/(n+1)^2}{1\big/n^2}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^2 = 1.$
- Therefore the ratio test is inconclusive here.

The Root Test

Root Test Let $\sum_{n=1}^{\infty} a_n$ be a series (the terms may be positive or negative).

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence is convergent).
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root test is inconclusive and we cannot determine if the series converges or diverges using this test.

Example 5 Test the following series for convergence $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n}{n+1}\right)^n$

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- Therefore by the n th root test, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n}{n+1}\right)^n$ diverges.

Example 6

Root Test For $\sum_{n=1}^{\infty} a_n$. $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.
If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
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- Therefore by the n th root test, the series $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$ converges.

Example 7

Root Test For $\sum_{n=1}^{\infty} a_n$. $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.
If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
If $L > 1$ or ∞, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
If $L = 1$, then the Root test is inconclusive.

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- Therefore by the n th root test, the series $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$ converges.

Rearranging sums

If we rearrange the terms in a finite sum, the sum remains the same. This is not always the case for infinite sums (infinite series). It can be shown that:

- If a series $\sum a_n$ is an absolutely convergent series with $\sum a_n = s$, then any rearrangement of $\sum a_n$ is convergent with sum s.
- It a series $\sum a_n$ is a conditionally convergent series, then for any real number r, there is a rearrangement of $\sum a_n$ which has sum r.

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- **Example** The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ is absolutely convergent with $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \frac{2}{3}$ and hence any rearrangement of the terms has sum $\frac{2}{3}$.

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$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \cdots + (-1)^n \frac{1}{n} + \cdots = \ln 2.$$

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- Now we rearrange the terms taking the positive terms in blocks of one followed by negative terms in blocks of 2

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$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots =$$

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- Obviously, we could continue in this way to get the series to sum to any number of the form $(\ln 2)/2^n$.