

Power Series

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where x is a variable, the c_n 's are constants called the coefficients of the series.

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- ▶ We see that a power series can converge for some values of x and diverge for others.
- ▶ In this case the series $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$ and diverges if $|x| \geq 1$, since it is a geometric series with $r = x$.

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Example Lets see what we can say about the power series

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- ▶ Note $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$. This series is always a geometric series with $r = \frac{x}{2}$.

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- ▶ Therefore this series converges when $|x| < 2$ and diverges if $|x| \geq 2$.
- ▶ Since this series converges only on the interval $-2 < x < 2$, we say that the **Interval of Convergence** of this series is the interval $(-2, 2)$.

Power series as functions

A **power series defines a function**

$$f(x) = c_0 + c_1x + c_2x^2 + \dots$$

whose **domain** is the set of all values of x for which the series converges.

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- ▶ We cannot always get a formula for a function defined by a power series. We will focus on that problem in subsequent lectures. Today we focus on finding the domain (Interval of convergence) of a power series.

Power Series Centered at a .

Definition A power series in $(x - a)$ or a **power series centered at a** is a power series of the form

$$\sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \dots$$

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► **Note** that when $x = a$, we have

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- **Note** also that when $a = 0$, the power series about a above just becomes a power series about 0 similar to the power series in our original definition and the previous examples.
- That is the power series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ is a power series centered around $a = 0$.

Example

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- We have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1} / (3^{n+1}(n+2)^3)}{|x-1|^n / (3^n(n+1)^3)}$
 $= \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1}}{|x-1|^n} \frac{3^n(n+1)^3}{3^{n+1}(n+2)^3} = \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1}}{|x-1|^n} \frac{3^n}{3^{n+1}} \frac{(n+1)^3}{(n+2)^3}$
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$$= \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1}}{|x-1|^n} \frac{3^n(n+1)^3}{3^{n+1}(n+2)^3} = \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1}}{|x-1|^n} \frac{3^n}{3^{n+1}} \frac{(n+1)^3}{(n+2)^3}$$

$$= \lim_{n \rightarrow \infty} \frac{|x-1|}{3} \left(\frac{n+1}{n+2} \right)^3 = \frac{|x-1|}{3} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^3 = \frac{|x-1|}{3}.$$

- ▶ Since we are using the ratio test, we conclude that the series converges if $\frac{|x-1|}{3} < 1$, it diverges if $\frac{|x-1|}{3} > 1$ and the test is inconclusive when $\frac{|x-1|}{3} = 1$.

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- ▶ We have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1} / (3^{n+1}(n+2)^3)}{|x-1|^n / (3^n(n+1)^3)}$

$$= \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1}}{|x-1|^n} \frac{3^n(n+1)^3}{3^{n+1}(n+2)^3} = \lim_{n \rightarrow \infty} \frac{|x-1|^{n+1}}{|x-1|^n} \frac{3^n}{3^{n+1}} \frac{(n+1)^3}{(n+2)^3}$$

$$= \lim_{n \rightarrow \infty} \frac{|x-1|}{3} \left(\frac{n+1}{n+2} \right)^3 = \frac{|x-1|}{3} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^3 = \frac{|x-1|}{3}.$$

- ▶ Since we are using the ratio test, we conclude that the series converges if $\frac{|x-1|}{3} < 1$, it diverges if $\frac{|x-1|}{3} > 1$ and the test is inconclusive when $\frac{|x-1|}{3} = 1$.
- ▶ That is, the series converges if $|x-1| < 3$, that is $-3 < x-1 < 3$. Adding 1 to both sides, we see this is equivalent to $-2 < x < 4$.

Example

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- ▶ For any given value of x , we apply the ratio test to the series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}$.
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- ▶ When $x = -2$, the series becomes $\sum_{n=0}^{\infty} \frac{(-2-1)^n}{3^n(n+1)^3} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n+1)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3}$. This series converges absolutely, since $\sum_{n=0}^{\infty} \frac{1}{(n+1)^3}$ converges by the limit comparison test, comparing with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

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- ▶ When $x = 4$, the series becomes $\sum_{n=0}^{\infty} \frac{(4-1)^n}{3^n(n+1)^3} = \sum_{n=0}^{\infty} \frac{(3)^n}{3^n(n+1)^3} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^3}$ which converges by the limit comparison test, comparing with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

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- ▶ Therefore, this series converges for x in the closed interval $[-2, 4]$ and diverges otherwise.

Radius of Convergence, Interval of Convergence

Theorem For any power series $\sum_{n=0}^{\infty} c_n(x - a)^n$, there are only 3 possibilities for the values of x for which the series converges :

1. The series converges only when $x = a$.
2. The series converges for all x .
3. There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. (In previous example $R = 3$, series converged when $|x - 1| < 3$ and diverged when $|x - 1| > 3$.)

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► **Definition** The **Radius of convergence (R.O.C.)** of the power series is the number R in case 3 above.

In case 1, the R.O.C. is 0 and in case 2, the R.O.C. is ∞ .

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► **Definition** The **Radius of convergence (R.O.C.)** of the power series is the number R in case 3 above.

In case 1, the R.O.C. is 0 and in case 2, the R.O.C. is ∞ .

► We see that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges within some interval centered at a and diverges outside that interval. The **Interval of Convergence (I.O.C.)** of a power series is the interval that consists of all values of x for which the series converges.

- In case 1 above, the interval of convergence is a single point $\{a\}$,
- In case 2 above the interval of convergence is $(-\infty, \infty)$.
- In case 3 above the interval of convergence may be

$$(a - R, a + R), [a - R, a + R), (a - R, a + R], [a - R, a + R].$$

Previous Example

Example The power series below is centered at 1.

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}.$$

We used the ratio test to determine that the series converges when $|x-1| < 3$ and diverges when $|x-1| > 3$. Therefore the radius of convergence of this series is 3.

We checked the endpoints of the interval $(-2, 4)$ using the limit comparison test to find that the series converges on the interval $[-2, 4]$ and diverges otherwise.

Therefore the interval of convergence for this series is $[-2, 4]$.

Example

Example Find the interval of convergence and radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

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$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

- ▶ We use the ratio test to determine where the series converges.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{\left| \frac{x^{n+1}}{(n+1)!} \right|}{\left| \frac{x^n}{n!} \right|} = \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{|x|^n} \frac{n!}{(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{|x|}{(n+1)} = |x| \lim_{n \rightarrow \infty} \frac{1}{(n+1)} = 0 \end{aligned}$$

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- ▶ This limit is zero for every value of x . This means that the power series converges for every value of x . Here the **radius of convergence** is $R = \infty$ and the **Interval of Convergence** is $(-\infty, \infty)$.

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- By the ratio test, this series converges if $|x| < 1$ and diverges if $|x| > 1$. The **Radius of Convergence** is $R = 1$.

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- ▶ To determine the I.O.C., we check the endpoints of the interval $|x| < 1$ or $-1 < x < 1$ giving x in $(-1, 1)$.

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- ▶ By the ratio test, this series converges if $|x| < 1$ and diverges if $|x| > 1$. The **Radius of Convergence** is $R = 1$.
- ▶ To determine the I.O.C., we check the endpoints of the interval $|x| < 1$ or $-1 < x < 1$ giving x in $(-1, 1)$.
- ▶ At $x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 1^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. This converges by the alternating series test, since $\lim_{n \rightarrow \infty} \frac{1}{2n+1} = 0$ and $\frac{1}{2n+1} > \frac{1}{2(n+1)+1}$ for all $n > 1$.

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- By the ratio test, this series converges if $|x| < 1$ and diverges if $|x| > 1$. The **Radius of Convergence** is $R = 1$.
- To determine the I.O.C., we check the endpoints of the interval $|x| < 1$ or $-1 < x < 1$ giving x in $(-1, 1)$.
- At $x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 1^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. This converges by the alternating series test, since $\lim_{n \rightarrow \infty} \frac{1}{2n+1} = 0$ and $\frac{1}{2n+1} > \frac{1}{2(n+1)+1}$ for all $n > 1$.
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- By the ratio test, this series converges if $|x| < 1$ and diverges if $|x| > 1$. The **Radius of Convergence** is $R = 1$.
- To determine the I.O.C., we check the endpoints of the interval $|x| < 1$ or $-1 < x < 1$ giving x in $(-1, 1)$.
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- Therefore the **Interval of Convergence** for this power series is $(-1, 1]$.

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Example Find the interval of convergence and radius of convergence of the following power series: $\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+1)4^n}$.

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$$\begin{aligned} \blacktriangleright \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{\left| \frac{(x+2)^{n+1}}{(n+2)4^{n+1}} \right|}{\left| \frac{(x+2)^n}{(n+1)4^n} \right|} = \lim_{n \rightarrow \infty} \frac{|x+2| \cdot (n+1)}{(n+2) \cdot 4} = \\ &= \frac{|x+2|}{4} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right) = \frac{|x+2|}{4}. \end{aligned}$$

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- ▶ The ratio test says that this power series converges if $\frac{|x+2|}{4} < 1$ or $|x+2| < 4$ and the series diverges if $|x+2| > 4$. The **Radius of Convergence** of this power series is $R = 4$.

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- ▶ To determine the I.O.C., we check the endpoints of the interval $|x+2| < 4$ or $-4 < x+2 < 4$ giving x in $(-6, 2)$.

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$$\begin{aligned} \blacktriangleright \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{\left| \frac{(x+2)^{n+1}}{(n+2)4^{n+1}} \right|}{\left| \frac{(x+2)^n}{(n+1)4^n} \right|} = \lim_{n \rightarrow \infty} \frac{|x+2| \cdot (n+1)}{(n+2) \cdot 4} = \\ &= \frac{|x+2|}{4} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right) = \frac{|x+2|}{4}. \end{aligned}$$

- $\blacktriangleright$ The ratio test says that this power series converges if $\frac{|x+2|}{4} < 1$ or $|x+2| < 4$ and the series diverges if $|x+2| > 4$. The **Radius of Convergence** of this power series is $R = 4$.
- $\blacktriangleright$ To determine the I.O.C., we check the endpoints of the interval $|x+2| < 4$ or $-4 < x+2 < 4$ giving x in $(-6, 2)$.
- $\blacktriangleright$ At $x = 2$: $\sum_{n=0}^{\infty} \frac{(2+2)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges by (limit) comparison with the harmonic series.

Example

Example Find the interval of convergence and radius of convergence of the following power series: $\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+1)4^n}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{\left| \frac{(x+2)^{n+1}}{(n+2)4^{n+1}} \right|}{\left| \frac{(x+2)^n}{(n+1)4^n} \right|} = \lim_{n \rightarrow \infty} \frac{|x+2| \cdot (n+1)}{(n+2) \cdot 4} = \\ &= \frac{|x+2|}{4} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right) = \frac{|x+2|}{4}. \end{aligned}$$

- ▶ The ratio test says that this power series converges if $\frac{|x+2|}{4} < 1$ or $|x+2| < 4$ and the series diverges if $|x+2| > 4$. The **Radius of Convergence** of this power series is $R = 4$.
- ▶ To determine the I.O.C., we check the endpoints of the interval $|x+2| < 4$ or $-4 < x+2 < 4$ giving x in $(-6, 2)$.
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- ▶ At $x = -6$: $\sum_{n=0}^{\infty} \frac{(-6+2)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by the alternating series test.

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Example Find the interval of convergence and radius of convergence of the following power series: $\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+1)4^n}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{\left| \frac{(x+2)^{n+1}}{(n+2)4^{n+1}} \right|}{\left| \frac{(x+2)^n}{(n+1)4^n} \right|} = \lim_{n \rightarrow \infty} \frac{|x+2| \cdot (n+1)}{(n+2) \cdot 4} = \\ &= \frac{|x+2|}{4} \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right) = \frac{|x+2|}{4}. \end{aligned}$$

- ▶ The ratio test says that this power series converges if $\frac{|x+2|}{4} < 1$ or $|x+2| < 4$ and the series diverges if $|x+2| > 4$. The **Radius of Convergence** of this power series is $R = 4$.
- ▶ To determine the I.O.C., we check the endpoints of the interval $|x+2| < 4$ or $-4 < x+2 < 4$ giving x in $(-6, 2)$.
- ▶ At $x = 2$: $\sum_{n=0}^{\infty} \frac{(2+2)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges by (limit) comparison with the harmonic series.
- ▶ At $x = -6$: $\sum_{n=0}^{\infty} \frac{(-6+2)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which converges by the alternating series test.
- ▶ Therefore the **Interval of Convergence** of this series is $[-6, 2)$.