Power Series representations of Functions

From our knowledge of Geometric Series, we know that

$$g(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1.$$

This gives us **a power series representation for the function $g(x)$ on the interval $(-1, 1)$.**
Note that the function $g(x)$ here has a larger domain than the power series.

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  and this series diverges for $|r| \geq 1$.
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- The n th partial sum of the above power series is given by $P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.
- Hence, as $n \to \infty$, the graphs of the polynomials, $P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$ get closer to the graph of $f(x)$ on the interval $(-1, 1)$.

Questions of the day

Q1. Given a function, can I find a power series representation of that function.

Part of the same question: If so for which values of x is the power series representation valid?

Q2 Why on earth would I want to write a nice function like that as a P.S.?

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Deriving new representations from old ones

Substitution First, we examine how to use the power series representation of the function $g(x) = 1/(1-x)$ on the interval $(-1, 1)$ to derive a power series representation of other functions on an interval.

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since we have $-1 < -x^7 < 1$ if $1 > x^7 > -1$ or $-1 < x^7 < 1$ or $-1 < x < 1$.

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- This is the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n x^{7n}$, since this series is easily seen to diverge at $x = 1$ and $x = -1$..

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- Since $f(x) = \frac{2x}{1+x} = 2x\frac{1}{1+x}$, we have
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- Alternatively in shorthand:

$$f(x) = \frac{2x}{1+x} = 2x\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty} 2x(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n 2x^{n+1}.$$

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- $h(x) = \frac{1}{4-x} = \frac{1}{4}\left[\frac{1}{1-\frac{x}{4}}\right].$
- With $y = \frac{x}{4}$, we get $\frac{1}{1-(\frac{x}{4})} = 1 + \left(\frac{x}{4}\right) + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3 + \cdots + \left(\frac{x}{4}\right)^n + \cdots = \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n \quad \text{for} \quad -1 < \left(\frac{x}{4}\right) < 1$

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- Therefore $\frac{1}{1-(\frac{x}{4})} = 1 + \frac{x}{4} + \frac{x^2}{4^2} + \frac{x^3}{4^3} + \cdots + \frac{x^4}{4^n} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{4^n} \quad \text{for} \quad -4 < x < 4$

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- We have $h(x) = \frac{1}{4}\left[\frac{1}{1-\frac{x}{4}}\right] =$ $\frac{1}{4}[1 + \frac{x}{4} + \frac{x^2}{4^2} + \frac{x^3}{4^3} + \cdots + \frac{x^4}{4^n} + \cdots] = \frac{1}{4}\sum_{n=0}^{\infty} \frac{x^n}{4^n}$ $= \frac{1}{4} + \frac{x}{4^2} + \frac{x^2}{4^3} + \frac{x^3}{4^4} + \cdots + \frac{x^4}{4^{n+1}} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}$

Differentiation and Integration of Power Series

We can differentiate and integrate power series term by term, just as we do with polynomials.

Theorem If the series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

Also

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_3\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of both of these power series is also R. (The interval of convergence may not remain the same when a series is differentiated or integrated; in particular convergence or divergence may change at the end points).

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- Now we multiply both sides by -1 to get
 $\frac{1}{(1+x)^2} = 0 + 1 - 2x + 3x^2 + \cdots + (-1)^{n+1} n x^{n-1} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} \quad \text{for} \quad -1 < x < 1$

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- integrating the left hand side and integrating the right hand side term by term, we get $\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty} \int (-1)^n x^n dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C \text{ for } \quad -1 < x < 1.$

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- Recall that if we have an alternating series which converges by the Alternating series test (the above series does), then if we estimate the sum of the series by adding the first M terms, the error $\left| \ln(1.1) - \sum_{n=0}^{M} (-1)^n \frac{(0.1)^{n+1}}{n+1} \right| = \left| \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{n+1}}{n+1} - \sum_{n=0}^{M} (-1)^n \frac{(0.1)^{n+1}}{n+1} \right| \leq |a_{M+1}|$

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- Therefore, if we use the estimate $\ln(1.1) \approx \sum_{n=0}^{3} (-1)^n \frac{(0.1)^{n+1}}{n+1} = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} = 0.1 - 0.005 + 0.000333 - 0.000025 = 0.0953083$, we must have an error less than or equal to $|a_{M+1}| = \left| \frac{(0.1)^5}{5} \right| < 10^{-5}$.

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- If we use a computer to check, we see that $|\ln(1.1) - 0.0953083| = 1.8798 \times 10^-6$.

Approximation of definite Integrals

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- This is an alternating series which sums to the definite integral $\int_0^{0.1} \frac{1}{1+x^7}\,dx$. I can estimate the sum of the series by taking a partial sum $S_n = a_0 + a_1 + a_2 + \cdots + a_n$ and the error of approximation is less than or equal to $|a_{n+1}|$.

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- Since $\left|\frac{(.1)^{15}}{15}\right| < (.1)^{15} < (.1)^{10}$, we must have that $\left| \int_0^{0.1} \frac{1}{1+x^7} dx \right| \approx (.1) - \frac{(.1)^8}{8} = .1000000125$ and the error of approximation is less than 10^{-10}.

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- Since $\tan^{-1}(0) = 0$, we have $C = 0$ and

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  $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$ on the interval $(-1,1)$.
- Now we can integrate term by term to get a power series representation of $\tan^{-1}(x)$ on the interval $(-1,1)$,

$$\tan^{-1}(x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots \quad \text{for} \quad x \in (-1,1).$$

- Since $\tan^{-1}(0) = 0$, we have $C = 0$ and

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots \quad \text{on the interval} \quad (-1,1).$$

- $\frac{1}{\sqrt{3}} < 1$ and with $x = \frac{1}{\sqrt{3}}$, we get

$$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{3}^{(2n+1)}(2n+1)}.$$