Applications of Taylor Series

Recall that we used the linear approximation of a function in Calculus 1 to estimate the values of the function near a point a (assuming f was differentiable at a):

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for} \quad x \quad \text{near} \quad a.$$

Now suppose that $f(x)$ has infinitely many derivatives at a and $f(x)$ equals the sum of its Taylor series in an interval around a, then we can approximate the values of the function $f(x)$ near a by the nth partial sum of the Taylor series at x, or the nth Taylor Polynomial:

$$f(x) \approx T_n(x)$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$T_n(x)$ is a polynomial of degree n with the property that $T_n(a) = f(a)$ and $T_n^{(i)}(a) = f^{(i)}(a)$ for $i = 1, 2, \ldots, n$.
Note that $T_1(x)$ is the linear approximation given above.

Example

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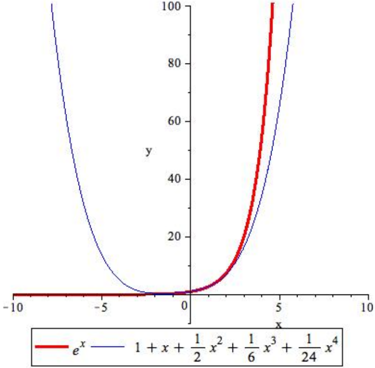

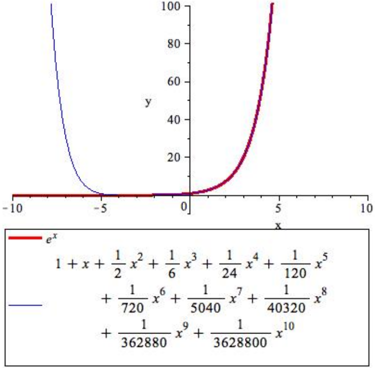

Approximations

If $f(x)$ equals the sum of its Taylor series (about a) at x, then we have

$$\lim_{n\to\infty} T_n(x) = f(x)$$

and larger values of n should give of better approximations to $f(x)$. The approximation We can use Taylor's Inequality to help estimate the error in our approximation.

The error in our approximation of $f(x)$ by $T_n(x)$ is $|R_n(x)| = |f(x) - T_n(x)|$. We can estimate the size of this error in two ways:

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- **1. Taylor's Inequality** If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$ then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality

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- **2.** If the Taylor series is an alternating series, we can use the alternating series estimate for the error.

Example

Example (a) Consider the approximation to the function $f(x) = e^x$ by the fourth McLaurin polynomial of $f(x)$ given above.

(b) How accurate is the approximation when $-4 \leq x \leq 4$? (Give an upper bound for the error on this interval).

(c) Find an interval around 0 for which this approximation has error $< .001$.

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Example (a) Consider the approximation to the function $f(x) = e^x$ by the fourth McLaurin polynomial of $f(x)$ given above.

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- If $|x| < 4$, Taylor's inequality says that $|R_n(x)| \leq \frac{e^4}{(5)!}|x|^5 < \frac{e^4}{(5)!}|4|^5 = 465.9$ on this interval.

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- If we assume that $r < 1$, we have $e^r < e$ and we need an r with $\frac{e}{(5)!}|r|^5 \leq .001$ or $|r|^5 < \frac{.001 \times 5!}{e}$. This works if $r < \sqrt[5]{\frac{.001 \times 5!}{e}} \approx 0.53$

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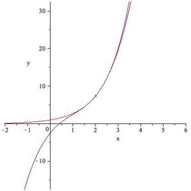
Example (a) Find the third Taylor polynomial of $f(x) = e^x$ at $a = 2$.



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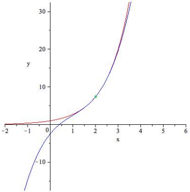

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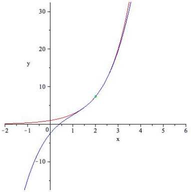

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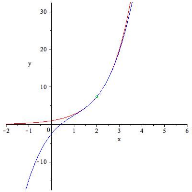

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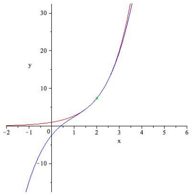

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- $M = e^3$ works and hence the error of approximation $= |R_n(x)| \leq \frac{e^3|x-2|^4}{4!} \leq \frac{e^3}{4!} = .837$ for any x in $(-1, 1)$.

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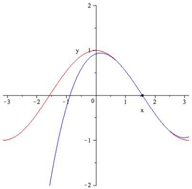

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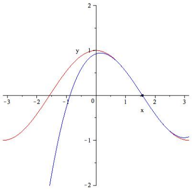

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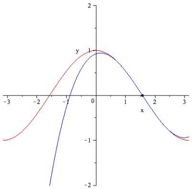

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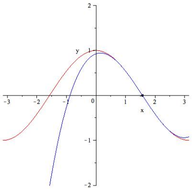

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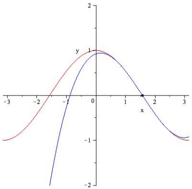

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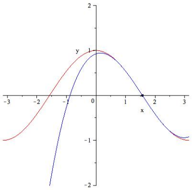

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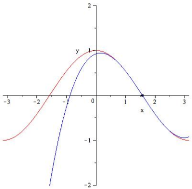

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- Therefore the error from the above approximation is
  $|R_n(x)| = |\cos x - T_3(x)| \leq \left|\frac{(x-\frac{\pi}{2})^5}{5!}\right| \leq \frac{\left(\frac{\pi}{4}\right)^5}{5!} = \frac{\pi^5}{4^5 5!} = .0024.$