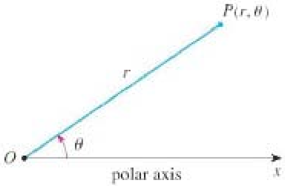
Polar Co-ordinates

A polar coordinate system, gives the co-ordinates of a point with reference to a point O and a half line or ray starting at the point O. We will look at polar coordinates for points in the xy-plane, using the origin $(0, 0)$ and the positive x-axis for reference.



A point P in the plane, has polar coordinates (r, θ), where r is the distance of the point from the origin and θ is the angle that the ray $|OP|$ makes with the positive x-axis.

Example 1

Example 1 Plot the points whose polar coordinates are given by

$$(2, \frac{\pi}{4}) \quad (3, -\frac{\pi}{4}) \quad (3, \frac{7\pi}{4}) \quad (2, \frac{5\pi}{2})$$

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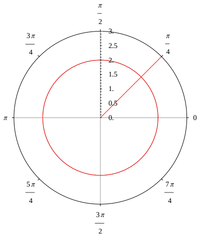

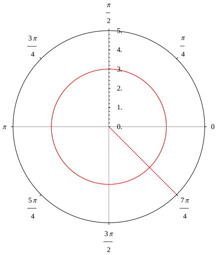

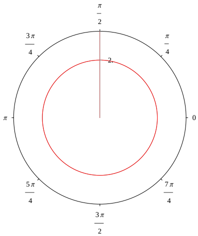

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Note the representation of a point in polar coordinates is not unique. For instance for any θ the point $(0, \theta)$ represents the pole O. We extend the meaning of polar coordinate to the case when r is negative by agreeing that the two points (r, θ) and $(-r, \theta)$ are in the same line through O and at the same distance $|r|$ but on opposite side of O. Thus

$$(-r, \theta) = (r, \theta + \pi)$$

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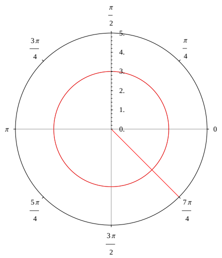
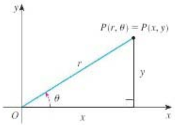


Polar to Cartesian coordinates

To convert from Polar to Cartesian coordinates, we use the identities:

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta}$$



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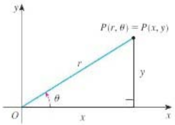

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- For $(2, \frac{\pi}{4})$, we have $r = 2$, $\theta = \frac{\pi}{4}$. In Cartesian co-ordinates, we get $x = r\cos\theta = 2\cos(\pi/4) = 2\frac{1}{\sqrt{2}} = \sqrt{2}$
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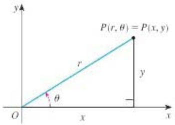

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- For $(3, -\frac{\pi}{3})$, we have $r = 3$, $\theta = -\frac{\pi}{3}$. In Cartesian co-ordinates, we get $x = r\cos\theta = 3\cos(-\frac{\pi}{3}) = 3\frac{1}{2} = \frac{3}{2}$
  we get $y = r\sin\theta = 3\sin(-\frac{\pi}{3}) = 3\frac{-\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2}$

Cartesian to Polar coordinates

To convert from Cartesian to polar coordinates, we use the following identities

$$\boxed{r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}}$$

When choosing the value of θ, we must be careful to consider which quadrant the point is in, since for any given number a, there are two angles with $\tan\theta = a$, in the interval $0 \leq \theta \leq 2\pi$.

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Example 3 Give polar coordinates for the points (given in Cartesian co-ordinates) $(2,2)$, $(1, -\sqrt{3})$, and $(-1, \sqrt{3})$

- For $(2,2)$, we have $x = 2$, $y = 2$. Therefore $r^2 = x^2 + y^2 = 4 + 4 = 8$, and $r = \sqrt{8}$.
  We have $\tan\theta = \frac{y}{x} = 2/2 = 1$.
  Since this point is in the first quadrant, we have $\theta = \frac{\pi}{4}$.
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- For $(1, -\sqrt{3})$, we have $x = 1$, $y = -\sqrt{3}$. Therefore $r^2 = x^2 + y^2 = 1 + 3 = 4$, and $r = 2$.
  We have $\tan\theta = \frac{y}{x} = -\sqrt{3}$.
  Since this point is in the fourth quadrant, we have $\theta = \frac{-\pi}{3}$.
  Therefore the polar co-ordinates for the point $(1, -\sqrt{3})$ are $(2, \frac{-\pi}{3})$.

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- For $(-1, \sqrt{3})$, we have $x = -1$, $y = \sqrt{3}$. Therefore $r^2 = x^2 + y^2 = 1 + 3 = 4$, and $r = 2$.
  We have $\tan\theta = \frac{y}{x} = -\sqrt{3}$.
  Since this point is in the second quadrant, we have $\theta = \frac{2\pi}{3}$.
  Therefore the polar co-ordinates for the point $(-1, \sqrt{3})$ are $(2, \frac{2\pi}{3})$.

Graphing Equations in Polar Coordinates

The graph of an equation in polar coordinates $r = f(\theta)$ or $F(r, \theta) = 0$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

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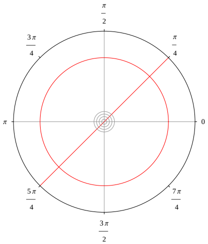

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- We find the value of r for specific values of θ in order to plot some points on the curve. We note that it is enough to sketch the graph on the interval $0 \leq \theta \leq 2\pi$, since $(r(\theta), \theta) = (r(\theta + 2\pi), \theta + 2\pi)$.

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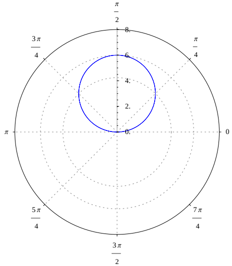
- When we plot the points, we see that they lie on a circle of radius 3 centered at $(0, 3)$ and that the curve traces out this circle twice in an anticlockwise direction as θ increases from 0 to 2π.



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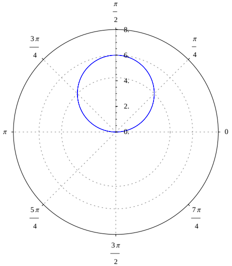

- The equation in Cart. co-ords is $x^2 + (y - 3)^2 = 9$.

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$\frac{3\pi}{4}$	$6/\sqrt{2}$	$(-3, 3)$
π	0	$(0, 0)$
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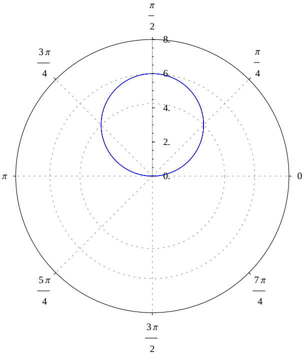

It may help to calculate the cartesian co-ordinates in order to sketch the curve.

Example 6

Example 6 Graph the equation $r = 1 + \cos\theta$. Check the variations shown at end of lecture notes.

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0		
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$\frac{\pi}{2}$		
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π		
$\frac{5\pi}{4}$		
$\frac{3\pi}{2}$		
$\frac{7\pi}{4}$		
2π		

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θ	r	Cartesian Co-ords (x, y)
0	2	$(2, 0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}+1}{\sqrt{2}}$	$(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2})$
$\frac{\pi}{2}$	1	$(0, 1)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}-1}{\sqrt{2}}$	$(\frac{-(\sqrt{2}-1)}{2}, \frac{\sqrt{2}-1}{2})$
π	0	$(0, 0)$
$\frac{5\pi}{4}$	$\frac{\sqrt{2}-1}{\sqrt{2}}$	$(\frac{-(\sqrt{2}-1)}{2}, \frac{-(\sqrt{2}-1)}{2})$
$\frac{3\pi}{2}$	1	$(0, -1)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}+1}{\sqrt{2}}$	$(\frac{(\sqrt{2}+1)}{2}, \frac{-(\sqrt{2}+1)}{2})$
2π	2	$(2, 0)$

- We find the value of r for specific values of θ in order to plot some points on the curve. We note that it is enough to sketch the graph on the interval

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π	0	$(0, 0)$
$\frac{5\pi}{4}$	$\frac{\sqrt{2}-1}{\sqrt{2}}$	$(\frac{-(\sqrt{2}-1)}{2}, \frac{-(\sqrt{2}-1)}{2})$
$\frac{3\pi}{2}$	1	$(0, -1)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}+1}{\sqrt{2}}$	$(\frac{(\sqrt{2}+1)}{2}, \frac{-(\sqrt{2}+1)}{2})$
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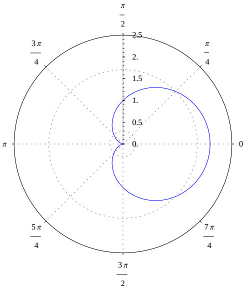
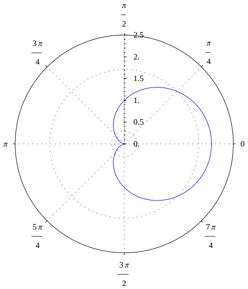
- When we plot the points, we see that they lie on a heart shaped curve.



Using Symmetry

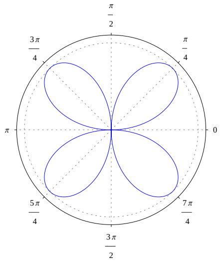
We have 3 common symmetries in curves which often shorten our work in graphing a curve of the form $r = f(\theta)$:

- If $f(-\theta) = f(\theta)$ the curve is symmetric about the horizontal line $\theta = 0$. In this case it is enough to draw either the upper or lower half of the curve, drawing the other half by reflecting in the line $\theta = 0$(the x-axis). [Example $r = 1 + \cos\theta$].



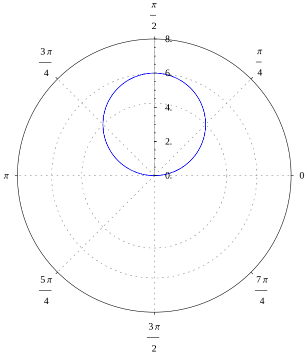
Using Symmetry

- If $f(\theta) = f(\theta + \pi)$, the curve has central symmetry about the pole or origin. Here it is enough to draw the graph in either the upper or right half plane and then rotate by 180 degree to get the other half. [Example $r = \sin 2\theta$.]



Using Symmetry

- If $f(\theta) = f(\pi - \theta)$, the curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$. It is enough to draw either the right half or the left half of the curve in this case. [Example $r = 6\sin\theta$.]



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Example 7 Sketch the **rose** $r = \cos(4\theta)$

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$\frac{\pi}{8}$	0
$\frac{\pi}{4}$	-1
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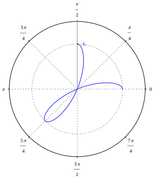
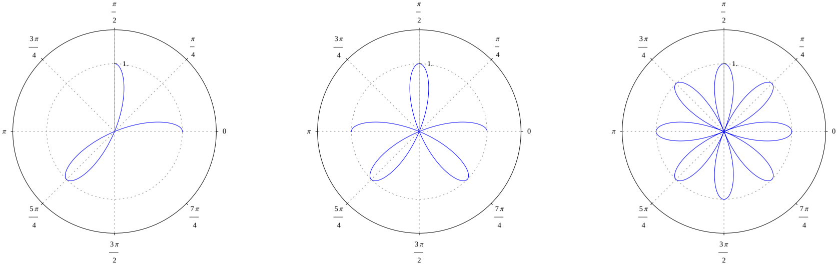


Example7

Example 7 Sketch the **rose** $r = \cos(4\theta)$

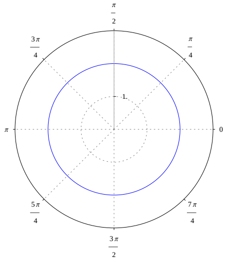
- Since $\cos(4\theta) = \cos(4(-\theta))$, we have symmetry with respect to the x-axis.
- Since $\cos(4\theta) = \cos(4(\pi - \theta)) = \cos(4\pi - 4\theta)$, we have symmetry with respect to the x-axis.
- Therefore it is enough to draw part of the graph for $0 \leq \theta \leq \frac{\pi}{2}$ and find the rest of the graph (for $0 \leq \theta \leq 2\pi$ or $-\pi \leq \theta \leq \pi$) by symmetry.

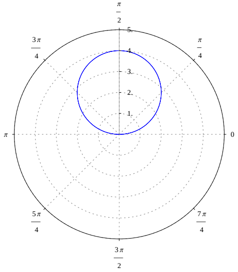


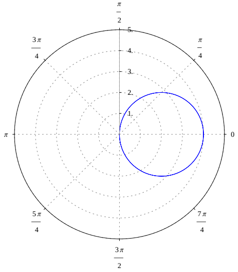
-
- We reflect in the y-axis to get the graph for $0 \leq \theta \leq \pi$ and subsequently in the x axis to get the graph for $-\pi \leq \theta \leq \pi$.

Circles

We have many equations of circles with polar coordinates: $r = a$ is the circle centered at the origin of radius a, $r = 2a\sin\theta$ is the circle of radius a centered at $(a, \frac{\pi}{2})$ (on the y-axis), and $r = 2a\cos\theta$ is the circle of radius a centered at $(a, 0)$ (on the x-axis).
Below, we show the graphs of $r = 2$, $r = 4\sin\theta$ and $r = 4\cos\theta$.







Tangents to Polar Curves

If we want to find the equation of a tangent line to a curve of the form $r = f(\theta)$, we write the equation of the curve in parametric form, using the parameter θ.

$$\boxed{x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta}$$

From the calculus of parametric equations, we know that if f is differentiable and continuous we have the formula:

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Note As usual, we locate horizontal tangents by identifying the points where $dy/dx = 0$ and we locate vertical tangents by identifying the points where $dy/dx = \infty$

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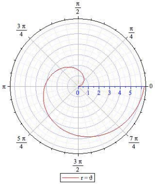
Example: Tangents to Polar Curves

Example 8 Find the equation of the tangent to the curve $r = \theta$ when $\theta = \frac{\pi}{2}$



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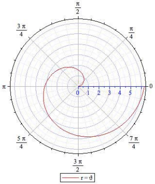

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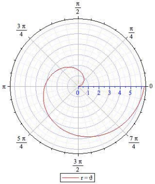

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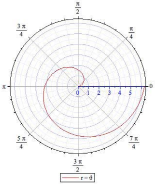

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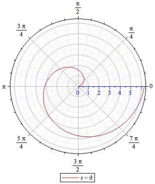

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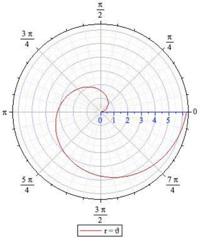

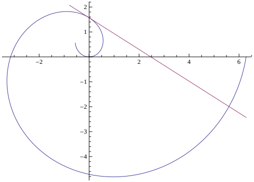

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- Therefore the equation of the tangent line, when $\theta = \frac{\pi}{2}$ is given by $(y - \frac{\pi}{2}) = -\frac{2}{\pi}x$.

Example 9

(a) Find the equation of the tangent to the circle $r = 6\sin\theta$ when $\theta = \frac{\pi}{3}$

(b) At which points do we have a horizontal tangent?

(c) At which points do we have a vertical tangent?

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- The tangent is vertical when $dx/d\theta = 0$ (as long as $dy/d\theta \neq 0$).
- This happens if $\cos^2\theta = \sin^2\theta$, which is true if $\cos\theta = \pm\sin\theta$. True if $\theta = (2n+1)\frac{\pi}{4}$.

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