Lengths in Polar Coordinates

Given a polar curve $r = f(\theta)$, we can use the relationship between Cartesian coordinates and Polar coordinates to write **parametric equations** which describe the curve using the parameter θ

$$x(\theta) = f(\theta)\cos\theta \qquad y(\theta) = f(\theta)\sin\theta$$

To compute the arc length of such a curve between $\theta = a$ and $\theta = b$, we need to compute the integral

$$L = \int_a^b \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$$

We can simplify this formula because
$(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2$
$= [f'(\theta)]^2 + [f(\theta)]^2 = r^2 + (\frac{dr}{d\theta})^2$ Thus

$$L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

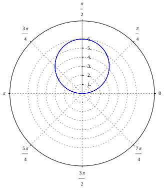
Example 1

Compute the length of the polar curve $r = 6\sin\theta$ for $0 \leq \theta \leq \pi$

Example 1

Compute the length of the polar curve $r = 6\sin\theta$ for $0 \le \theta \le \pi$

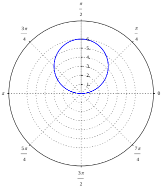
- Last day, we saw that the graph of this equation is a circle of radius 3 and as θ increases from 0 to π, the curve traces out the circle exactly once.



Example 1

Compute the length of the polar curve $r = 6\sin\theta$ for $0 \le \theta \le \pi$

- Last day, we saw that the graph of this equation is a circle of radius 3 and as θ increases from 0 to π, the curve traces out the circle exactly once.

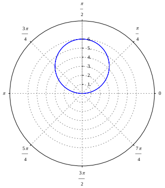


- We have $L = \int_0^{\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$

Example 1

Compute the length of the polar curve $r = 6\sin\theta$ for $0 \leq \theta \leq \pi$

- Last day, we saw that the graph of this equation is a circle of radius 3 and as θ increases from 0 to π, the curve traces out the circle exactly once.

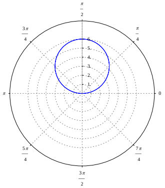


- We have $L = \int_0^{\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$
- $r = 6\sin\theta$ and $\frac{dr}{d\theta} = 6\cos\theta$.

Example 1

Compute the length of the polar curve $r = 6\sin\theta$ for $0 \le \theta \le \pi$

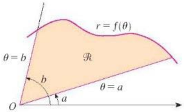
- Last day, we saw that the graph of this equation is a circle of radius 3 and as θ increases from 0 to π, the curve traces out the circle exactly once.

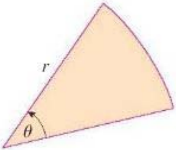


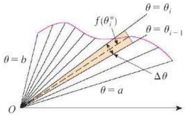
- We have $L = \int_0^{\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$
- $r = 6\sin\theta$ and $\frac{dr}{d\theta} = 6\cos\theta$.
- $L = \int_0^{\pi} \sqrt{36\sin^2\theta + 36\cos^2\theta} d\theta = \int_0^{\pi} 6\sqrt{\sin^2\theta + \cos^2\theta} d\theta = 6\pi$.

Areas in Polar Coordinates

Suppose we are given a polar curve $r = f(\theta)$ and wish to calculate the area swept out by this polar curve between two given angles $\theta = a$ and $\theta = b$. This is the region $\mathcal{R}$ in the picture on the left below:







Dividing this shape into smaller pieces (on right) and estimating the areas of the small pieces with pie-like shapes (center picture), we get a Riemann sum approximating $A =$ the area of $\mathcal{R}$ of the form

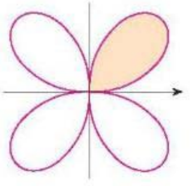
$$A \approx \sum_{i=1}^{n} \frac{f(\theta_i^*)^2}{2} \Delta\theta$$

Letting $n \to \infty$ yields the following integral expression of the area

$$\boxed{\int_a^b \frac{f(\theta)^2}{2} d\theta = \int_a^b \frac{r^2}{2} d\theta}$$

Example 2

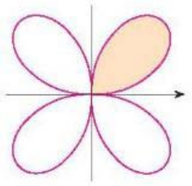
Compute the area bounded by the curve $r = \sin 2\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$.



$r = \sin 2\theta$

Example 2

Compute the area bounded by the curve $r = \sin 2\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$.

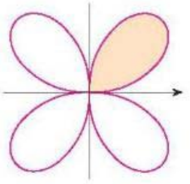


$r = \sin 2\theta$

- Using the formula $A = \int_a^b \frac{f(\theta)^2}{2} d\theta = \int_0^{\frac{\pi}{2}} \frac{(\sin 2\theta)^2}{2} d\theta$

Example 2

Compute the area bounded by the curve $r = \sin 2\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$.



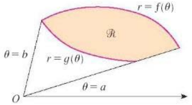
$r = \sin 2\theta$

- Using the formula $A = \int_a^b \frac{f(\theta)^2}{2} d\theta = \int_0^{\frac{\pi}{2}} \frac{(\sin 2\theta)^2}{2} d\theta$
- Using the half-angle formula, we get $A = \frac{1}{4} \int_0^{\frac{\pi}{2}} 1 - \cos(4\theta) d\theta$ $= \frac{1}{4} \left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\pi/2} = \frac{\pi}{8}$.

Areas of Region between two curves

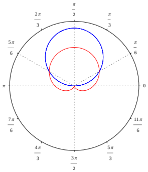
If instead we consider a region bounded between two polar curves $r = f(\theta)$ and $r = g(\theta)$ then the equations becomes

$$\boxed{\frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta}$$



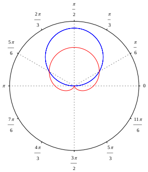
Example 3

Example 3 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



Example 3

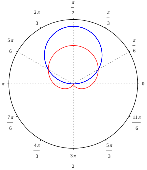
Example 3 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



- We use the formula $A = \frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta$

Example 3

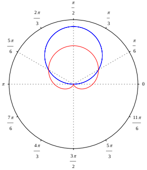
Example 3 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



- We use the formula $A = \frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta$
- To find the angles where the curves intersect, we find the values of θ where $3\sin\theta = 1 + \sin\theta$ or $2\sin\theta = 1$. That is $\sin\theta = 1/2$

Example 3

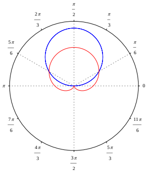
Example 3 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



- We use the formula $A = \frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta$
- To find the angles where the curves intersect, we find the values of θ where $3\sin\theta = 1 + \sin\theta$ or $2\sin\theta = 1$. That is $\sin\theta = 1/2$
- Therefore $\theta = \frac{\pi}{6}, \ \frac{5\pi}{6}$.

Example 3

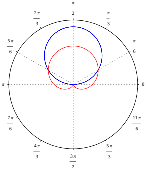
Example 3 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



- We use the formula $A = \frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta$
- To find the angles where the curves intersect, we find the values of θ where $3\sin\theta = 1 + \sin\theta$ or $2\sin\theta = 1$. That is $\sin\theta = 1/2$
- Therefore $\theta = \frac{\pi}{6}, \ \frac{5\pi}{6}$.
- $\frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 9\sin^2\theta - (1 + \sin\theta)^2 d\theta$

Example 3

Example 3 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

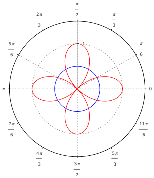


- We use the formula $A = \frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta$
- To find the angles where the curves intersect, we find the values of θ where $3\sin\theta = 1 + \sin\theta$ or $2\sin\theta = 1$. That is $\sin\theta = 1/2$
- Therefore $\theta = \frac{\pi}{6},\ \frac{5\pi}{6}$.
- $\frac{1}{2}\int_a^b f(\theta)^2 - g(\theta)^2 d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 9\sin^2\theta - (1 + \sin\theta)^2 d\theta$
- $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 8\sin^2\theta - 1 - 2\sin\theta d\theta = 4\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2\theta d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 d\theta - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin\theta d\theta$
  $= 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 - \cos(2\theta) d\theta - \frac{\pi}{3} + \cos\theta\Big]_{\pi/6}^{5\pi/6} = \frac{4\pi}{3} - \sin(2\theta)\Big]_{\pi/6}^{5\pi/6} - \frac{\pi}{3} + \sqrt{3}$
  $= \pi$.

Word of Warning

NOTE The fact that a single point has many representation in polar coordinates makes it very difficult to find all the points of intersections of two polar curves. It is important to draw the two curves!!!

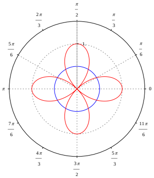
Example 4 Find all possible intersection points of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.



Word of Warning

NOTE The fact that a single point has many representation in polar coordinates makes it very difficult to find all the points of intersections of two polar curves. It is important to draw the two curves!!!

Example 4 Find all possible intersection points of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

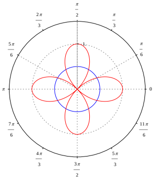


- Here we must solve two equations $\frac{1}{2} = \cos(2\theta)$ and $-\frac{1}{2} = \cos(2\theta)$ since the curve $r = \frac{1}{2}$ is the same as the curve $r = -\frac{1}{2}$.

Word of Warning

NOTE The fact that a single point has many representation in polar coordinates makes it very difficult to find all the points of intersections of two polar curves. It is important to draw the two curves!!!

Example 4 Find all possible intersection points of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

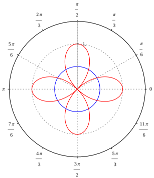


- Here we must solve two equations $\frac{1}{2} = \cos(2\theta)$ and $-\frac{1}{2} = \cos(2\theta)$ since the curve $r = \frac{1}{2}$ is the same as the curve $r = -\frac{1}{2}$.
- From the first equation, we get $2\theta = \pm\frac{\pi}{3}, \ \pm\frac{5\pi}{3}$ and from the second we get $2\theta = \pm\frac{2\pi}{3}, \quad \pm\frac{4\pi}{3}$.

Word of Warning

NOTE The fact that a single point has many representation in polar coordinates makes it very difficult to find all the points of intersections of two polar curves. It is important to draw the two curves!!!

Example 4 Find all possible intersection points of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.



- Here we must solve two equations $\frac{1}{2} = \cos(2\theta)$ and $-\frac{1}{2} = \cos(2\theta)$ since the curve $r = \frac{1}{2}$ is the same as the curve $r = -\frac{1}{2}$.
- From the first equation, we get $2\theta = \pm\frac{\pi}{3}, \ \pm\frac{5\pi}{3}$ and from the second we get $2\theta = \pm\frac{2\pi}{3}, \quad \pm\frac{4\pi}{3}$.
- Therefore we have 8 points of intersection $\theta = \pm\frac{\pi}{6}, \quad \pm\frac{\pi}{3}, \quad \pm\frac{2\pi}{3}, \quad \pm\frac{5\pi}{6}$.