General Logarithms and Exponentials

Last day, we looked at the inverse of the logarithm function, the exponential function. we have the following formulas:

$\boxed{\ln(x)}$	$\boxed{e^x}$
$\ln(ab) = \ln a + \ln b, \quad \ln(\frac{a}{b}) = \ln a - \ln b$	$\ln e^x = x \text{ and } e^{\ln(x)} = x$
$\ln a^x = x \ln a$	$e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^y = e^{xy}.$
$\lim_{x \to \infty} \ln x = \infty, \quad \lim_{x \to 0} \ln x = -\infty$	$\lim_{x \to \infty} e^x = \infty, \text{ and } \lim_{x \to -\infty} e^x = 0$
$\frac{d}{dx} \ln \lvert x \rvert = \frac{1}{x}$	$\frac{d}{dx} e^x = e^x$
$\int \frac{1}{x} dx = \ln \lvert x \rvert + C$	$\int e^x dx = e^x + C$

General exponential functions

For $a > 0$ and x any real number, we define

$$\boxed{a^x = e^{x \ln a}, \quad a > 0.}$$

The function a^x is called the exponential function with base a.
Note that $\ln(a^x) = x \ln a$ is true for all real numbers x and all $a > 0$. (We saw this before for x a rational number).
Note: The above definition for a^x does not apply if $a < 0$.

Laws of Exponents

We can derive the following laws of exponents directly from the definition and the corresponding laws for the exponential function e^x:

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

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- The other laws follow in a similar manner.

Derivatives

We can also derive the following rules of differentiation using the definition of the function a^x, $a > 0$, the corresponding rules for the function e^x and the chain rule.

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = a^x \ln a \qquad \frac{d}{dx}(a^{g(x)}) = \frac{d}{dx}e^{g(x)\ln a} = g'(x)a^{g(x)} \ln a$$

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- Example: Find the derivative of 5^{x^3+2x}.
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- $= (\ln 5)(3x^2 + 2) e^{(x^3+2x) \ln 5} = (\ln 5)(3x^2 + 2) 5^{x^3+2x}$.

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- We get a different graph for each possible value of a. We split the analysis into two cases,

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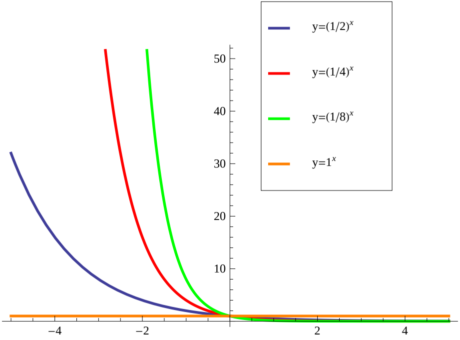

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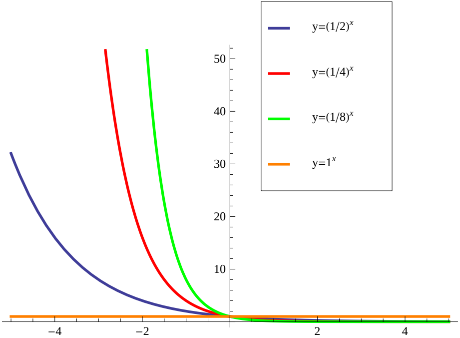

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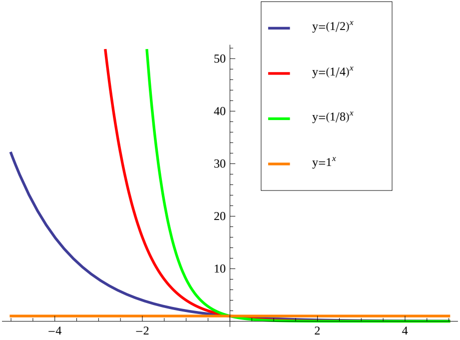

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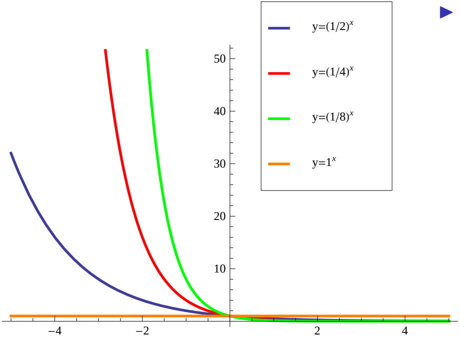

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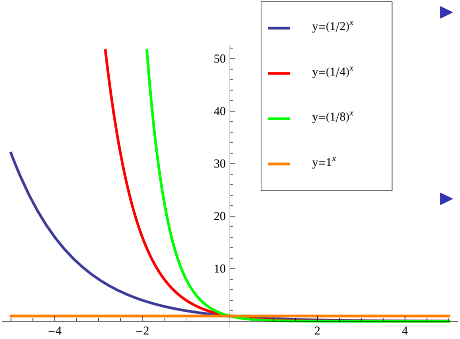

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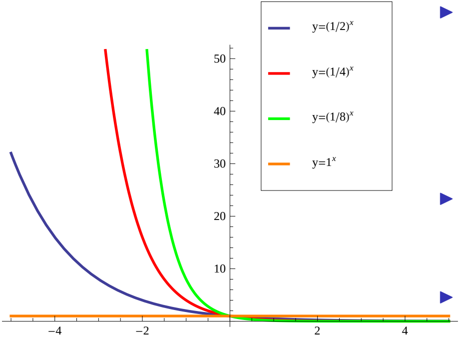

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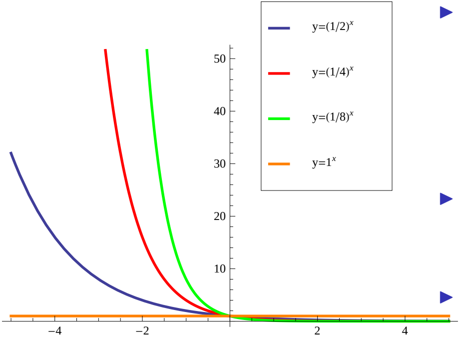

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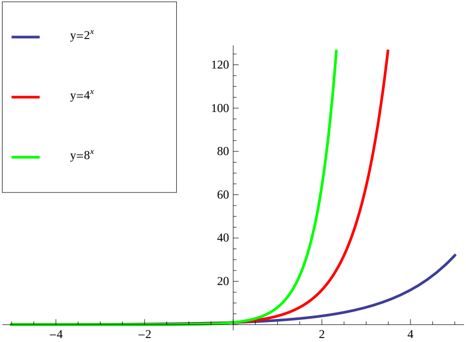

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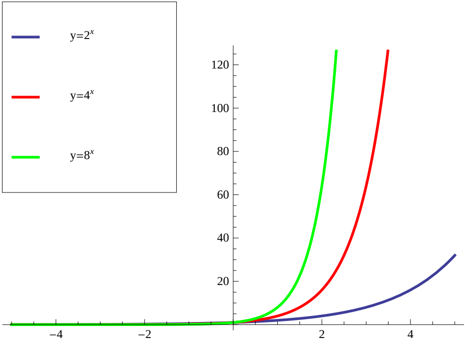

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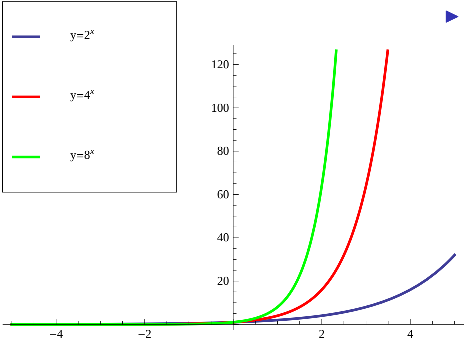

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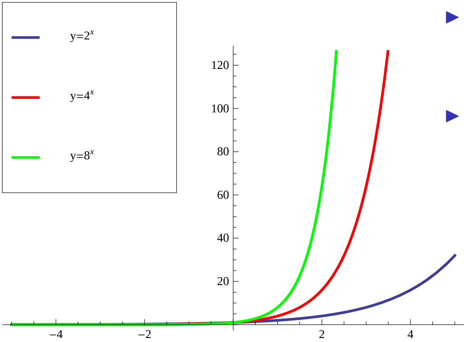

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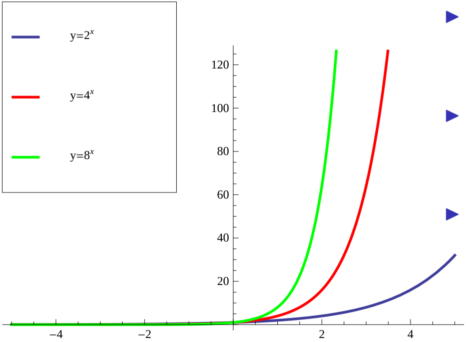

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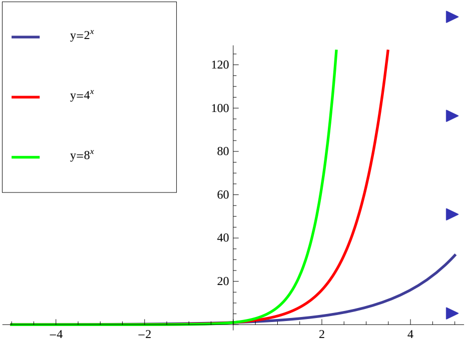

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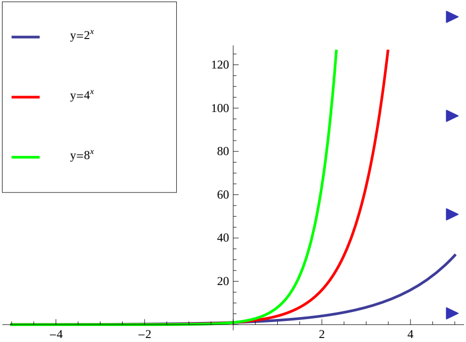

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For $a > 1$, $\lim_{x \to \infty} a^x = \infty$, $\lim_{x \to -\infty} a^x = 0$.

Power Rules

We now have 4 different types of functions involving bases and powers. So far we have dealt with the first three types:
If a and b are constants and $g(x) > 0$ and $f(x)$ and $g(x)$ are both differentiable functions.

$$\frac{d}{dx}a^b = 0, \qquad \frac{d}{dx}(f(x))^b = b(f(x))^{b-1}f'(x), \qquad \frac{d}{dx}a^{g(x)} = g'(x)a^{g(x)}\ln a,$$

$$\frac{d}{dx}(f(x))^{g(x)}$$

For $\frac{d}{dx}(f(x))^{g(x)}$, we use **logarithmic differentiation or write the function as** $(f(x))^{g(x)} = e^{g(x)\ln(f(x))}$ **and use the chain rule**.

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- Also to calculate limits of functions of this type it may help write the function as $(f(x))^{g(x)} = e^{g(x)\ln(f(x))}$.

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- $$f(f^{-1}(x)) = x \qquad f^{-1}(f(x)) = x$$
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Change of base Formula

It is not difficult to show that $\log_a x$ has similar properties to $\ln x = \log_e x$. This follows from the **Change of Base Formula** which shows that The function $\log_a x$ is a constant multiple of $\ln x$.

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$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

- Let $y = \log_a x$.
- Since a^x is the inverse of $\log_a x$, we have $a^y = x$.
- Taking the natural logarithm of both sides, we get $y \ln a = \ln x$,

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- which gives, $y = \frac{\ln x}{\ln a}$.
- The algebraic properties of the natural logarithm thus extend to general logarithms, by the change of base formula.

$$\log_a 1 = 0, \qquad \log_a(xy) = \log_a(x) + \log_a(y), \qquad \log_a(x^r) = r \log_a(x).$$

for any positive number $a \neq 1$. In fact for most calculations (especially limits, derivatives and integrals) it is advisable to convert $\log_a x$ to natural logarithms. The most commonly used logarithm functions are $\log_{10} x$ and $\ln x = \log_e x$.

Using Change of base Formula for derivatives

Change of base formula

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

From the above change of base formula for $\log_a x$, we can easily derive the following **differentiation formulas**:

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{x \ln a} \qquad \frac{d}{dx}(\log_a g(x)) = \frac{g'(x)}{g(x)\ln a}.}$$

A special Limit

We derive the following limit formula by taking the derivative of $f(x) = \ln x$ at $x = 1$, We know that $f'(1) = 1/1 = 1$. We also know that

$$f'(1) = \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \ln(1+x)^{1/x} = 1.$$

Applying the (continuous) exponential function to the limit on the left hand side (of the last equality), we get

$$e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1+x)^{1/x}.$$

Applying the exponential function to the right hand sided(of the last equality), we gat $e^1 = e$. Hence

$$\boxed{e = \lim_{x \to 0} (1+x)^{1/x}}$$

Note If we substitute $y = 1/x$ in the above limit we get

$$\boxed{e = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y \quad \text{and} \quad e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

where n is an integer (see graphs below). We look at large values of n below to get an approximation of the value of e.

A special Limit

$$n = 10 \rightarrow \left(1 + \frac{1}{n}\right)^n = 2.59374246, \quad n = 100 \rightarrow \left(1 + \frac{1}{n}\right)^n = 2.70481383,$$

$$n = 100 \rightarrow \left(1 + \frac{1}{n}\right)^n = 2.71692393, \quad n = 1000 \rightarrow \left(1 + \frac{1}{n}\right)^n = 2.1814593.$$