General Logarithms and Exponentials

Last day, we looked at the inverse of the logarithm function, the exponential function. We have the following formulas:

\[
\ln(ab) = \ln a + \ln b, \quad \ln\left(\frac{a}{b}\right) = \ln a - \ln b
\]

\[
\ln(a^x) = x \ln a
\]

\[
\lim_{x \to \infty} \ln x = \infty, \quad \lim_{x \to 0} \ln x = -\infty
\]

\[
\frac{d}{dx} \ln |x| = \frac{1}{x}
\]

\[
\int \frac{1}{x} dx = \ln |x| + C
\]

\[
e^x = e^x \quad \text{and} \quad e^{\ln(x)} = x
\]

\[
e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^y = e^{xy}
\]

\[
\lim_{x \to \infty} e^x = \infty, \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0
\]

\[
\frac{d}{dx} e^x = e^x
\]

\[
\int e^x dx = e^x + C
\]
For $a > 0$ and $x$ any real number, we define

$$a^x = e^{x \ln a}, \quad a > 0.$$  

The function $a^x$ is called the exponential function with base $a$. Note that $\ln(a^x) = x \ln a$ is true for all real numbers $x$ and all $a > 0$. (We saw this before for $x$ a rational number).  

**Note:** The above definition for $a^x$ does not apply if $a < 0$.  

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We can derive the following laws of exponents directly from the definition and the corresponding laws for the exponential function $e^x$:

$$a^{x+y} = a^x a^y \quad a^{x-y} = \frac{a^x}{a^y} \quad (a^x)^y = a^{xy} \quad (ab)^x = a^x b^x$$
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For example, we can prove the first rule in the following way:
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a^{x+y} = e^{(x+y)\ln a}
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Laws of Exponents

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- For example, we can prove the first rule in the following way:
  - $a^{x+y} = e^{(x+y) \ln a}$
  - $= e^{x \ln a + y \ln a}$
  - $= a^x a^y$
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    a^{x+y} &= e^{(x+y)\ln a} \\
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\]

► The other laws follow in a similar manner.
We can also derive the following rules of differentiation using the definition of the function $a^x$, $a > 0$, the corresponding rules for the function $e^x$ and the chain rule.

$$
\frac{d}{dx} (a^x) = \frac{d}{dx} (e^{x \ln a}) = a^x \ln a
$$

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\frac{d}{dx} (a^{g(x)}) = \frac{d}{dx} (e^{g(x) \ln a}) = g'(x) a^{g(x)} \ln a
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- Example: Find the derivative of $5^{x^3+2x}$.
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Example: Find the derivative of $5^{x^3+2x}$.

Instead of memorizing the above formulas for differentiation, I can just convert this to an exponential function of the form $e^{h(x)}$ using the definition of $5^u$, where $u = x^3 + 2x$ and differentiate using the techniques we learned in the previous lecture.
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We have, by definition, $5^{x^3+2x} = e^{(x^3+2x) \ln 5}$
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- Therefore $\frac{d}{dx} 5^{x^3+2x} = \frac{d}{dx} e^{(x^3+2x) \ln 5} = e^{(x^3+2x) \ln 5} \frac{d}{dx} (x^3 + 2x) \ln 5$
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$= (\ln 5)(3x^2 + 2)e^{(x^3+2x) \ln 5} = (\ln 5)(3x^2 + 2)5^{x^3+2x}$.
For $a > 0$ we can draw a picture of the graph of

$$y = a^x$$

using the techniques of graphing developed in Calculus I.

- We get a different graph for each possible value of $a$. We split the analysis into two cases,
Graphs of General exponential functions

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- since the family of functions $y = a^x$ slope downwards when $0 < a < 1$ and
- the family of functions $y = a^x$ slope upwards when $a > 1$. 

Case 1: Graph of $y = a^x$, $0 < a < 1$

- **y-intercept**: The y-intercept is given by $y = a^0 = e^0 = 1$.
- **x-intercept**: The values of $a^x = e^x \ln a$ are always positive and there is no x-intercept.
- **Slope**: If $0 < a < 1$, the graph of $y = a^x$ has a negative slope and is always decreasing. $\frac{d}{dx}(a^x) = a^x \ln a < 0$. In this case, a smaller value of $a$ gives a steeper curve for $x < 0$.
- The graph is concave up since the second derivative is $\frac{d^2}{dx^2}(a^x) = a^x (\ln a)^2 > 0$.
- As $x \to \infty$, $x \ln a$ approaches $-\infty$, since $\ln a < 0$ and therefore $a^x = e^x \ln a \to 0$.
- As $x \to -\infty$, $x \ln a$ approaches $\infty$, since both $x$ and $\ln a$ are less than 0. Therefore $a^x = e^x \ln a \to \infty$.

For $0 < a < 1$, $\lim_{x \to \infty} a^x = 0$, $\lim_{x \to -\infty} a^x = \infty$.
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\[\lim_{x \to \infty} a^x = 0, \quad \lim_{x \to -\infty} a^x = \infty\]
Case 2: Graph of $y = a^x$, $a > 1$

- **y-intercept:** The y-intercept is given by $y = a^0 = e^{0 \ln a} = e^0 = 1$. 

- If $a > 1$, the graph of $y = a^x$ has a positive slope and is always increasing, $\frac{dy}{dx}(a^x) = a^x \ln a > 0$. 

- The graph is concave up since the second derivative is $\frac{d^2y}{dx^2}(a^x) = a^x (\ln a)^2 > 0$. 

- In this case a larger value of $a$ gives a steeper curve when $x > 0$. 

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Therefore $a^x = e^{x \ln a} \to 0$. For $a > 1$, $\lim_{x \to \infty} a^x = \infty$, $\lim_{x \to -\infty} a^x = 0$. 

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For $a > 1$, $\lim_{x \to \infty} ax = \infty$, $\lim_{x \to -\infty} ax = 0$. 

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\[
\begin{align*}
\text{For } a > 1, \quad & x \to \infty \quad a^x = \infty, \quad x \to -\infty \quad a^x = 0.
\end{align*}
\]
We now have 4 different types of functions involving bases and powers. So far we have dealt with the first three types:
If \( a \) and \( b \) are constants and \( g(x) > 0 \) and \( f(x) \) and \( g(x) \) are both differentiable functions.

\[
\frac{d}{dx} a^b = 0, \quad \frac{d}{dx} (f(x))^b = b(f(x))^{b-1}f'(x), \quad \frac{d}{dx} a^{g(x)} = g'(x)a^{g(x)} \ln a,
\]

\[
\frac{d}{dx} (f(x))^{g(x)}
\]

For \( \frac{d}{dx} (f(x))^{g(x)} \), we use logarithmic differentiation or write the function as \( (f(x))^{g(x)} = e^{g(x) \ln(f(x))} \) and use the chain rule.

**Example** Differentiate \( x^{2x^2} , x > 0 \).
Power Rules

We now have 4 different types of functions involving bases and powers. So far we have dealt with the first three types:

If \( a \) and \( b \) are constants and \( g(x) > 0 \) and \( f(x) \) and \( g(x) \) are both differentiable functions.

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\frac{d}{dx} a^b = 0, \quad \frac{d}{dx} (f(x))^b = b(f(x))^{b-1}f'(x), \quad \frac{d}{dx} a^g(x) = g'(x)a^{g(x)} \ln a,
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**Example** Differentiate \( x^{2x^2}, x > 0 \).

▶ Also to calculate limits of functions of this type it may help write the function as \( (f(x))^{g(x)} = e^{g(x) \ln(f(x))} \).
General Logarithmic Functions

Since \( f(x) = a^x \) is a monotonic function whenever \( a \neq 1 \), it has an inverse which we denote by

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f^{-1}(x) = \log_a x.
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- We get the following from the properties of inverse functions:

\[
f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x
\]

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\log_a (x) = y \quad \text{if and only if} \quad a^y = x
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We get the following from the properties of inverse functions:

1. \( f^{-1}(x) = y \) if and only if \( f(y) = x \)
2. \( \log_a(x) = y \) if and only if \( a^y = x \)
3. \( f(f^{-1}(x)) = x \) \( f^{-1}(f(x)) = x \)
4. \( a^{\log_a(x)} = x \) \( \log_a(a^x) = x \).
It is not difficult to show that $\log_a x$ has similar properties to $\ln x = \log_e x$. This follows from the **Change of Base Formula** which shows that the function $\log_a x$ is a constant multiple of $\ln x$.

\[
\log_a x = \frac{\ln x}{\ln a}
\]
Change of base Formula

It is not difficult to show that \( \log_a x \) has similar properties to \( \ln x = \log_e x \). This follows from the Change of Base Formula which shows that the function \( \log_a x \) is a constant multiple of \( \ln x \).

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\log_a x = \frac{\ln x}{\ln a}
\]

Let \( y = \log_a x \).
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- Let $y = \log_a x$.
- Since $a^y$ is the inverse of $\log_a x$, we have $a^y = x$. 

Annette Pilkington

**Natural Logarithm and Natural Exponential**
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$$\log_a x = \frac{\ln x}{\ln a}$$

- Let $y = \log_a x$.
- Since $a^y$ is the inverse of $\log_a x$, we have $a^y = x$.
- Taking the natural logarithm of both sides, we get $y \ln a = \ln x$,
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- Taking the natural logarithm of both sides, we get \( y \ln a = \ln x \),
- which gives, \( y = \frac{\ln x}{\ln a} \).
- The algebraic properties of the natural logarithm thus extend to general logarithms, by the change of base formula.

\[
\log_a 1 = 0, \quad \log_a(xy) = \log_a(x) + \log_a(y), \quad \log_a(x^r) = r \log_a(x)
\]

for any positive number \( a \neq 1 \). In fact for most calculations (especially limits, derivatives and integrals) it is advisable to convert \( \log_a x \) to natural logarithms. The most commonly used logarithm functions are \( \log_{10} x \) and \( \ln x = \log_e x \). Annette Pilkington

Natural Logarithm and Natural Exponential
Using Change of base Formula for derivatives

Change of base formula

\[ \log_a x = \frac{\ln x}{\ln a} \]

From the above change of base formula for \( \log_a x \), we can easily derive the following **differentiation formulas**:

\[
\frac{d}{dx} (\log_a x) = \frac{d}{dx} \frac{\ln x}{\ln a} = \frac{1}{x \ln a} \\
\frac{d}{dx} (\log_a g(x)) = \frac{g'(x)}{g(x) \ln a}.
\]
A special Limit

We derive the following limit formula by taking the derivative of \( f(x) = \ln x \) at \( x = 1 \). We know that \( f'(1) = 1/1 = 1 \). We also know that

\[
f'(1) = \lim_{x \to 0} \frac{\ln(1 + x) - \ln 1}{x} = \lim_{x \to 0} \ln(1 + x)^{1/x} = 1.
\]

Applying the (continuous) exponential function to the limit on the left hand side (of the last equality), we get

\[
e^{\lim_{x \to 0} \ln(1 + x)^{1/x}} = \lim_{x \to 0} e^{\ln(1 + x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}.
\]

Applying the exponential function to the right hand side (of the last equality), we get \( e^1 = e \). Hence

\[
e = \lim_{x \to 0} (1 + x)^{1/x}
\]

Note If we substitute \( y = 1/x \) in the above limit we get

\[
e = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y \quad \text{and} \quad e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n
\]

where \( n \) is an integer (see graphs below). We look at large values of \( n \) below to get an approximation of the value of \( e \).
A special Limit

\[ n = 10 \rightarrow \left(1 + \frac{1}{n}\right)^n = 2.59374246, \quad n = 100 \rightarrow \left(1 + \frac{1}{n}\right)^n = 2.70481383, \]

\[ n = 100 \rightarrow \left(1 + \frac{1}{n}\right)^n = 2.71692393, \quad n = 1000 \rightarrow \left(1 + \frac{1}{n}\right)^n = 2.1814593. \]