Solutions to the Differential Equation $\frac{dy(t)}{dt} = ky(t)$

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- **Interest** *If we invest \$$A_0$ in an account paying $r \times 100$ % interest per anumn and the interest is compounded continuously, the amount in the account after t years is given by*

$$\boxed{A(t) = A_0 e^{rt}.}$$

Interest Compounded Continuously

Example If I invest \$1000 for 5 years at a 4% interest rate with the interest compounded continuously,

(a) how much will be in my account at the end of the 5 years?

(b) How long before there is \$2000 in the account?

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(b) How long before there is \$2000 in the account?

- *We must solve for t in the equation* $2000 = 1000e^{0.04t}$.
- *Dividing by 1000 and taking the natural logarithm of both sides, we get*

$$2 = e^{0.04t} \quad \rightarrow \quad \ln 2 = 0.04t \quad \rightarrow \quad t = \ln 2/0.04 \approx 17.33 yrs.$$

Compound Interest

Sometimes interest is not compounded continuously. If I invest $\$A_0$ in an account with an interest rate of $r\%$ per annum, the amount in the bank account after t years depends on the number of times the interest is compounded per year. In the chart below

$A_0 = A(0)$ is the initial amount invested at time $t = 0$.
$A(t)$ is the amount in the account after t years.
$n =$ the number of times the interest is compounded per year.

We Have $$A(t) = A_0(1 + \frac{r}{n})^{nt}$$

Amt. after t years	$A(0)$	$A(1)$	$A(2)$	...	$A(t)$
$n = 1$	A_0	$A_0(1+r)$	$A_0(1+r)^2$	...	$A_0(1+r)^t$
$n = 2$	A_0	$A_0(1+\frac{r}{2})^2$	$A_0(1+\frac{r}{2})^4$	...	$A_0(1+\frac{r}{2})^{2t}$
$n = 12$	A_0	$A_0(1+\frac{r}{12})^{12}$	$A_0(1+\frac{r}{12})^{24}$	...	$A_0(1+\frac{r}{12})^{12t}$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	A_0	$A_0(1+\frac{r}{n})^n$	$A_0(1+\frac{r}{n})^{2n}$	...	$A_0(1+\frac{r}{n})^{nt}$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$n \to \infty$ (compounded continuously)	A_0 $= A_0$	$\lim_{n\to\infty} A_0(1+\frac{r}{n})^n$ $= A_0e^r$	$\lim_{n\to\infty} A_0(1+\frac{r}{n})^{2n}$ $= A_0e^{2r}$	... ...	$\lim_{n\to\infty} A_0(1+\frac{r}{n})^{nt}$ $= A_0e^{rt}$

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- $A(5) = 50,000(1 + \frac{.1}{4})^{20} \approx 81,930.82$

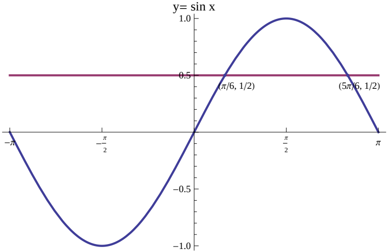
Restricted Sine Function.

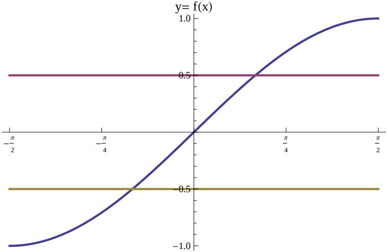
The trigonometric function $\sin x$ is not one-to-one functions, hence in order to create an inverse, we must restrict its domain.

The restricted sine function is given by

$$f(x) = \begin{cases} \sin x & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

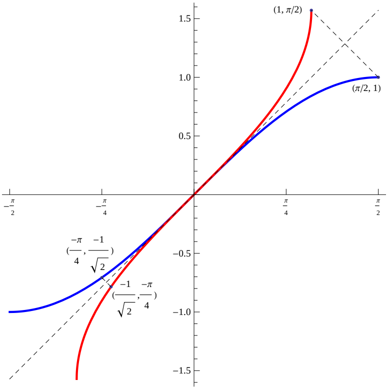
We have Domain(f) $= [-\frac{\pi}{2}, \frac{\pi}{2}]$ and Range(f) $= [-1, 1]$.





Inverse Sine Function (arcsin x $= sin^{-1}x$).

We see from the graph of the restricted sine function (or from its derivative) that the function is one-to-one and hence has an inverse, shown in red in the diagram below.



This inverse function, $f^{-1}(x)$, is denoted by $\boxed{f^{-1}(x) = \sin^{-1} x \quad \text{or} \quad \arcsin x.}$

Properties of $\sin^{-1} x$.

$$\text{Domain}(\sin^{-1}) = [-1, 1] \text{ and Range}(\sin^{-1}) = [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

Since $f^{-1}(x) = y$ if and only if $f(y) = x$, we have:

$$\boxed{\sin^{-1} x = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.}$$

Since $f(f^{-1})(x) = x$ $f^{-1}(f(x)) = x$ we have:

$$\boxed{\sin(\sin^{-1}(x)) = x \text{ for } x \in [-1, 1] \quad \sin^{-1}(\sin(x)) = x \text{ for } x \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big].}$$

from the graph: $\sin^{-1} x$ is an odd function and $\sin^{-1}(-x) = -\sin^{-1} x$.

Evaluating $\sin^{-1} x$.

Example Evaluate $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ using the graph above.

Example Evaluate $\sin^{-1}(\sqrt{3}/2)$ and $\sin^{-1}(-\sqrt{3}/2)$.

Evaluating $\sin^{-1} x$.

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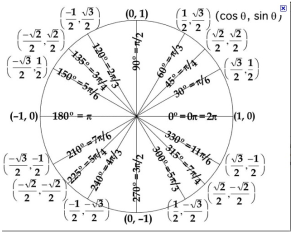

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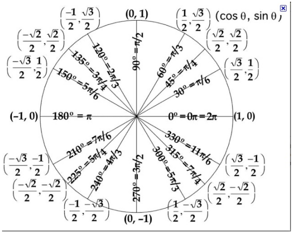

- $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\frac{\pi}{3}$

More Examples For $\sin^{-1} x$

Example Evaluate $\sin^{-1}(\sin \pi)$.

Example Evaluate $\cos(\sin^{-1}(\sqrt{3}/2))$.

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- *We saw above that* $\sin^{-1}(\sqrt{3}/2) = \frac{\pi}{3}$.
- *Therefore* $\cos(\sin^{-1}(\sqrt{3}/2)) = \cos\left(\frac{\pi}{3}\right) = 1/2$.

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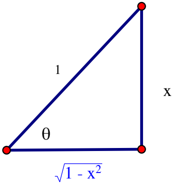

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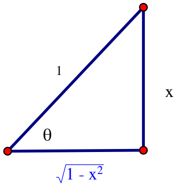

- *From this we see that* $\tan(\sin^{-1}(x)) = \tan(\theta) = \frac{x}{\sqrt{1-x^2}}$.

Derivative of $\sin^{-1} x$.

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad -1 \leq x \leq 1.}$$

Please read through the proof given in your notes using implicit differentiation. We can also derive a formula for $\frac{d}{dx}\sin^{-1}(k(x))$ using the chain rule, or we can apply the above formula along with the chain rule directly.

ExampleFind the derivative

$$\frac{d}{dx}\sin^{-1}\sqrt{\cos x}$$

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- $$= \frac{1}{\sqrt{1-\cos x}} \cdot \frac{-\sin x}{2\sqrt{\cos x}} = \frac{-\sin x}{2\sqrt{\cos x}\sqrt{1-\cos x}}$$

Inverse Cosine Function

We can define the function $\cos^{-1} x$ similarly. You can read the definition in your book. It can be shown that that $\frac{d}{dx} \cos^{-1} x = -\frac{d}{dx} \sin^{-1} x$ and one can use this to prove that

$$\boxed{\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}}.$$

Restricted Tangent Function

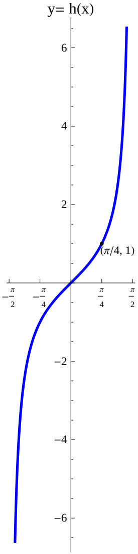
The tangent function is not a one to one function.
The restricted tangent function is given by

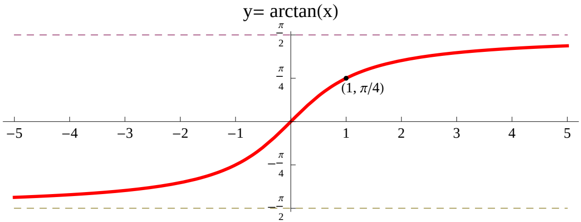
$$h(x) = \begin{cases} \tan x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We see from the graph of the restricted tangent function (or from its derivative) that the function is one-to-one and hence has an inverse, which we denote by

$$\boxed{h^{-1}(x) = \tan^{-1} x \quad \text{or} \quad \arctan x.}$$

Graphs of Restricted Tangent and $tan^{-1}x$.





Properties of $tan^{-1}x$.

$$\text{Domain}(\tan^{-1}) = (-\infty, \infty) \text{ and Range}(\tan^{-1}) = (-\tfrac{\pi}{2}, \tfrac{\pi}{2}).$$

Since $h^{-1}(x) = y$ if and only if $h(y) = x$, we have:

$$\tan^{-1} x = y \text{ if and only if } \tan(y) = x \text{ and } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

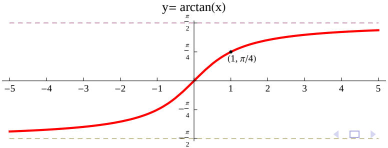
Since $h(h^{-1}(x)) = x$ and $h^{-1}(h(x)) = x$, we have:

$$\tan(\tan^{-1}(x)) = x \text{ for } x \in (-\infty, \infty) \quad \tan^{-1}(\tan(x)) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

From the graph, we have: $\tan^{-1}(-x) = -\tan^{-1}(x)$.

Also, since $\lim_{x \to (\frac{\pi}{2}-)} \tan x = \infty$ and $\lim_{x \to (-\frac{\pi}{2}+)} \tan x = -\infty$,

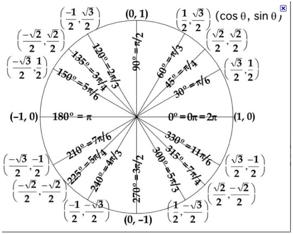
we have $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$ and $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$



Evaluating $\tan^{-1} x$

Example Find $\tan^{-1}(1)$ and $\tan^{-1}(\frac{1}{\sqrt{3}})$.

Example Find $\cos(\tan^{-1}(\frac{1}{\sqrt{3}}))$.



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- $\tan^{-1}(1)$ *is the unique angle,* θ*, between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *with* $\tan\theta = \frac{\sin\theta}{\cos\theta} = 1$*. By inspecting the unit circle, we see that* $\theta = \frac{\pi}{4}$*.*

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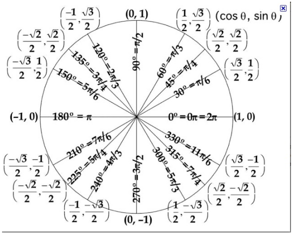

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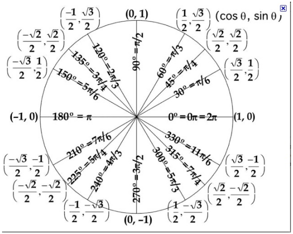

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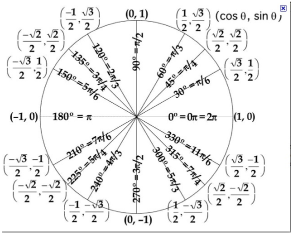

Derivative of $\tan^{-1} x$.

Using implicit differentiation, we get

$$\boxed{\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}, \quad -\infty < x < \infty.}$$

(Please read through the proof in your notes.) We can use the chain rule in conjunction with the above derivative.

Example Find the domain and derivative of $\tan^{-1}(\ln x)$

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- *Domain* $=$ *Domain of* $\ln x = (0, \infty)$
- $$\frac{d}{dx}\tan^{-1}(\ln x) = \frac{\frac{1}{x}}{1+(\ln x)^2} = \frac{1}{x(1+(\ln x)^2)}.$$

Integration Formulas

Reversing the derivative formulas above, we get

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C, \qquad \int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C,$$

Example

$$\int_0^{1/2} \frac{1}{1+4x^2}\,dx$$

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- $$\int_0^{1/2} \frac{1}{1+4x^2}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}u\Big|_0^1 = \frac{1}{2}[\tan^{-1}(1) - \tan^{-1}(0)]$$

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- $$\frac{1}{2}[\frac{\pi}{4} - 0] = \frac{\pi}{8}.$$

Integration

Example

$$\int \frac{1}{\sqrt{9 - x^2}}\, dx$$

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- *Let* $u = \frac{x}{3}$, *then* $dx = 3du$

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- *Let* $u = \frac{x}{3}$, *then* $dx = 3du$
- $$\int \frac{1}{\sqrt{9 - x^2}}\, dx = \frac{1}{3}\int \frac{3}{\sqrt{1 - u^2}} du = \sin^{-1} u + C = \sin^{-1}\frac{x}{3} + C$$

Indeterminate forms of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Definition An indeterminate form of the type $\frac{0}{0}$ is a limit of a quotient where both numerator and denominator approach 0.

Example

$$\lim_{x\to 0}\frac{e^x-1}{\sin x} \qquad \lim_{x\to\infty}\frac{x^{-2}}{e^{-x}} \qquad \lim_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}}$$

Definition An indeterminate form of the type $\frac{\infty}{\infty}$ is a limit of a quotient $\frac{f(x)}{g(x)}$ where $f(x)\to\infty$ or $-\infty$ and $g(x)\to\infty$ or $-\infty$.

Example

$$\lim_{x\to\infty}\frac{x^2+2x+1}{e^x} \qquad \lim_{x\to 0^+}\frac{x^{-1}}{\ln x}.$$

Indeterminate forms of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

L'Hospital's Rule Suppose *lim* stands for any one of

$$\lim_{x \to a} \qquad \lim_{x \to a^+} \qquad \lim_{x \to a^-} \qquad \lim_{x \to \infty} \qquad \lim_{x \to -\infty}$$

and $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
If *lim* $\frac{f'(x)}{g'(x)}$ is a finite number L or is $\pm\infty$, then

$$\boxed{lim \frac{f(x)}{g(x)} = lim \frac{f'(x)}{g'(x)}.}$$

(Assuming that $f(x)$ and $g(x)$ are both differentiable in some open interval around a or ∞ (as appropriate) except possible at a, and that $g'(x) \neq 0$ in that interval).

Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x\to 0} \frac{e^x - 1}{\sin x}$$

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- *Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L'Hospital's rule.*
- $$\lim_{x \to 0} \frac{e^x - 1}{\sin x} \underset{(L'Hosp.)}{=} \lim_{x \to 0} \frac{e^x}{\cos x} \underset{(Eval.)}{=} 1$$

Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}$$

Examples of Indeterminate forms of type $\frac{0}{0}$.

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Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}$$

- *Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L'Hospital's rule.*
- *As it stands, this quotient gets more complicated when we apply L'Hospital's rule, so we rearrange it before we apply the rule.*

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- $$\lim_{x\to\infty} \frac{x^{-2}}{e^{-x}} = \lim_{x\to\infty} \frac{1/x^2}{1/e^x} = \lim_{x\to\infty} \frac{e^x}{x^2}$$

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- $$\lim_{x\to\infty} \frac{e^x}{x^2} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{e^x}{2x} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{e^x}{2}$$

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- *As $x \to \infty$, we have $e^x \to \infty$ and therefore* $\lim_{x\to\infty} \frac{e^x}{2} = \infty$.

Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x \longrightarrow \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

- *Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L'Hospital's rule. ($\cos x$ and $x - \frac{\pi}{2}$ are both differentiable everywhere and $g'(x) \neq 0$ where $g(x) = x - \pi/2$).*

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- $$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} \underset{(L'Hosp.)}{=} \lim_{x \to \frac{\pi}{2}} \frac{-\sin x}{1} \underset{(Eval.)}{=} -1$$

Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

Example Find

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x}$$

Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

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- $$\lim_{x\to\infty} \frac{x^2+2x+1}{e^x} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{2x+2}{e^x} \underset{(L'Hosp.)}{=} \lim_{x\to\infty} \frac{2}{e^x}$$

Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

Example Find

$$\lim_{x\to\infty} \frac{x^2+2x+1}{e^x}$$

- *Since this is an indeterminate form of type* $\frac{\infty}{\infty}$, *we can apply L'Hospital's rule.*
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- *As* $x \to \infty$, *we have* $e^x \to \infty$ *and therefore* $\lim_{x\to\infty} \frac{2}{e^x} = 0$.

Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

Example Find

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Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

Example Find

$$\lim_{x \to 0^+} \frac{x^{-1}}{\ln(x)}$$

- *Since this is an indeterminate form of type $\frac{\infty}{\infty}$, we can apply L'Hospital's rule.*
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$$\lim_{x \to 0^+} \frac{x^{-1}}{\ln(x)} \underset{(L'Hosp.)}{=} \lim_{x \to 0^+} \frac{-x^{-2}}{1/x} = \lim_{x \to 0^+} \frac{-1/x^2}{1/x} = \lim_{x \to 0^+} \frac{-1}{x} = -\infty$$

Indeterminate forms of type $0 \cdot \infty$.

Definition $lim\ f(x)g(x)$ is **an indeterminate form of the type** $0 \cdot \infty$ if

$$lim\ f(x) = 0 \quad \text{and} \quad lim\ g(x) = \pm\infty.$$

Example $\lim_{x \to 0} x \ln |x|$

We can convert the above indeterminate form to an indeterminate form of type $\frac{0}{0}$ by writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)}$$

or to an indeterminate form of the type $\frac{\infty}{\infty}$ by writing

$$f(x)g(x) = \frac{g(x)}{1/f(x)}.$$

We them apply L'Hospital's rule to the limit.

Example of an Indeterminate form of type $0 \cdot \infty$.

Example $\lim_{x \to 0} x \ln |x|$

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- $\lim_{x \to 0} x \ln |x| = \lim_{x \to 0} \frac{\ln |x|}{1/x}$.

Example of an Indeterminate form of type $0 \cdot \infty$.

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- *We can convert the above indeterminate form to an indeterminate form of type* $\frac{\infty}{\infty}$ *by writing*
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- $\lim_{x \to 0} x \ln |x| = \lim_{x \to 0} \frac{\ln |x|}{1/x}$.
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- *We then apply L'Hospital's rule to the limit.*
-
$$\lim_{x \to 0} \frac{\ln |x|}{1/x} \underset{(L'Hosp.)}{=} = \lim_{x \to 0} \frac{1/x}{(-1/x^2)} = \lim_{x \to 0} \frac{1}{(-1/x)} = \lim_{x \to 0} (-x) = 0$$

Indeterminate forms of type 0^0, ∞^0, 1^∞.

Type	Limit		
0^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 0$	$lim\ g(x) = 0$
∞^0	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = \infty$	$lim\ g(x) = 0$
1^∞	$lim\ [f(x)]^{g(x)}$	$lim\ f(x) = 1$	$lim\ g(x) = \infty$

Example $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$.

Method

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Example $\lim_{x \to 0}(1 + x)^{\frac{1}{x}}$.

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- Look at $lim\ \ln[f(x)]^{g(x)} = lim\ g(x) \ln[f(x)]$.

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- Look at $lim\ \ln[f(x)]^{g(x)} = lim\ g(x)\ln[f(x)]$.
- Use L'Hospital to find $lim\ g(x)\ln[f(x)] = \alpha$. ($\alpha$ might be finite or $\pm\infty$ here.)

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0^0	$\lim\ [f(x)]^{g(x)}$	$\lim\ f(x) = 0$	$\lim\ g(x) = 0$
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Example $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$.

Method

- Look at $\lim\ \ln[f(x)]^{g(x)} = \lim\ g(x)\ln[f(x)]$.
- Use L'Hospital to find $\lim\ g(x)\ln[f(x)] = \alpha$. ($\alpha$ might be finite or $\pm\infty$ here.)
- Then $\lim\ f(x)^{g(x)} = \lim\ e^{\ln[f(x)]^{g(x)}} = e^\alpha$ since e^x is a continuous function. (where e^∞ should be interpreted as ∞ and $e^{-\infty}$ should be interpreted as 0.)

Example of an Indeterminate form of type 1^∞.

Example $\lim_{x \to 0}(1 + x)^{\frac{1}{x}}$.

Method

Example of an Indeterminate form of type 1^{∞}.

Example $\lim_{x \to 0}(1 + x)^{\frac{1}{x}}$.

Method

- *Look at* $\lim \ln[f(x)]^{g(x)} = \lim\ g(x) \ln[f(x)]$: *Look at* $\lim_{x \to 0} \ln[1 + x)]^{\frac{1}{x}} = \lim_{x \to 0} \frac{1}{x} \ln[1 + x]$

Example of an Indeterminate form of type 1^∞.

Example $\lim_{x \to 0}(1+x)^{\frac{1}{x}}$.

Method

- *Look at* $\lim \ln[f(x)]^{g(x)} = \lim g(x)\ln[f(x)]$*: Look at* $\lim_{x\to 0} \ln[1+x)]^{\frac{1}{x}} = \lim_{x \to 0} \frac{1}{x}\ln[1+x]$
- *Use L'Hospital to find* $\lim g(x)\ln[f(x)] = \alpha$.

$$\lim_{x\to 0} \frac{1}{x}\ln[1+x] = \lim_{x \to 0} \frac{\ln[1+x]}{x} \underset{(L'Hosp.)}{=} \lim_{x\to 0} \frac{1/[1+x]}{1} = 1 (= \alpha).$$

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Example $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$.

Method

- *Look at* $\lim \ln[f(x)]^{g(x)} = \lim g(x)\ln[f(x)]$: *Look at* $\lim_{x\to 0}\ln[1+x)]^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\ln[1+x]$
- *Use L'Hospital to find* $\lim g(x)\ln[f(x)] = \alpha$.

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- *Then* $\lim f(x)^{g(x)} = \lim e^{\ln[f(x)]^{g(x)}} = e^{\lim \ln[f(x)]^{g(x)}} = e^{\alpha}$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln[(1+x)^{\frac{1}{x}}]} = e^{\lim_{x\to 0}\ln[(1+x)^{\frac{1}{x}}]} = e^{1} = e.$$

Indeterminate forms of type $\infty - \infty$.

Indeterminate Forms of the type $\infty - \infty$ occur when we encounter a limit of the form
$lim(f(x) - g(x))$ where $lim\ f(x) = lim\ g(x) = \infty$ or
$lim\ f(x) = lim\ g(x) = -\infty$

To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...

Example $\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x}$

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- *This is an indeterminate form of type* $\frac{0}{0}$ *so we can use L'Hospital.*
- $$\lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \underset{(L'Hosp.)}{=} = \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$$
$$\underset{(L'Hosp.)}{=} = \lim_{x \to 0^+} \frac{-\sin x}{\cos x + (\cos x - x \sin x)} = \frac{0}{2} = 0$$