Solutions to the Differential Equation $\frac{dy(t)}{dt} = ky(t)$

Last Day, we saw that all solutions $y(t)$ to the differential equation $\frac{dy(t)}{dt} = ky(t)$ are of the form

$$y(t) = y(0)e^{kt}.$$

Such a function describes exponential growth when $k > 0$ and exponential decay when $k < 0$. Last day, we worked through examples of Population growth and radioactive decay.
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*This differential equation also applies to interest compounded continuously*

$$\frac{dA(t)}{dt} = rA(t), \quad A(t) = \text{amount in account at time } t, \ r = \text{interest rate (see below)}.$$
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- **This differential equation also applies to interest compounded continuously** 
  \[ \frac{dA(t)}{dt} = rA(t), \quad A(t) = \text{amount in account at time } t, \quad r = \text{interest rate} \ (\text{see below}) \]

- **Interest** If we invest $A_0$ in an account paying $r \times 100$ % interest per annum and the interest is compounded continuously, the amount in the account after $t$ years is given by

  $$A(t) = A_0e^{rt}.$$
Example If I invest $1000 for 5 years at a 4% interest rate with the interest compounded continuously,
(a) how much will be in my account at the end of the 5 years?

(b) How long before there is $2000 in the account?
Interest Compounded Continuously

Example If I invest $1000 for 5 years at a 4% interest rate with the interest compounded continuously,
(a) how much will be in my account at the end of the 5 years?

▶ We are given that $A_0 = 1000$ and $r = 0.04$.

(b) How long before there is $2000 in the account?
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- We are given that \( A_0 = 1000 \) and \( r = 0.04 \).
- Because the interest is compounded continuously, we have \( A(t) = A_0 e^{0.04t} = 1000e^{0.04t} \)

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\[
A(t) = A_0 e^{0.04t} = 1000e^{0.04t}
\]

▶ \( A(5) = 1000e^{0.04(5)} = $1221.4. \)

(b) How long before there is $2000 in the account?

▶ We must solve for \( t \) in the equation 
\[
2000 = 1000e^{0.04t}
\]

▶ Dividing by 1000 and taking the natural logarithm of both sides, we get 
\[
2 = e^{0.04t} \rightarrow \ln 2 = 0.04t \rightarrow t = \frac{\ln 2}{0.04} \approx 17.33 \text{ yrs}.
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2 = e^{0.04t} \quad \rightarrow \quad \ln 2 = 0.04t \quad \rightarrow \quad t = \ln 2 / 0.04 \approx 17.33 \text{ yrs}.
\]
Compound Interest

Sometimes interest is not compounded continuously. If I invest $A_0$ in an account with an interest rate of $r\%$ per annum, the amount in the bank account after $t$ years depends on the number of times the interest is compounded per year. In the chart below

\[ A_0 = A(0) \] is the initial amount invested at time $t = 0$.

\[ A(t) \] is the amount in the account after $t$ years.

\[ n \] is the number of times the interest is compounded per year.

We Have

\[ A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt} \]

<table>
<thead>
<tr>
<th>Amt. after $t$ years</th>
<th>$A(0)$</th>
<th>$A(1)$</th>
<th>$A(2)$</th>
<th>\ldots</th>
<th>$A(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1$</td>
<td>$A_0$</td>
<td>$A_0(1 + r)$</td>
<td>$A_0(1 + r)^2$</td>
<td>\ldots</td>
<td>$A_0(1 + r)^t$</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>$A_0$</td>
<td>$A_0\left(1 + \frac{r}{2}\right)^2$</td>
<td>$A_0\left(1 + \frac{r}{2}\right)^4$</td>
<td>\ldots</td>
<td>$A_0\left(1 + \frac{r}{2}\right)^{2t}$</td>
</tr>
<tr>
<td>$n = 12$</td>
<td>$A_0$</td>
<td>$A_0\left(1 + \frac{r}{12}\right)^{12}$</td>
<td>$A_0\left(1 + \frac{r}{12}\right)^{24}$</td>
<td>\ldots</td>
<td>$A_0\left(1 + \frac{r}{12}\right)^{12t}$</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>$n \to \infty$</td>
<td>$A_0$</td>
<td>$\lim_{n \to \infty} A_0\left(1 + \frac{r}{n}\right)^n$</td>
<td>$\lim_{n \to \infty} A_0\left(1 + \frac{r}{n}\right)^{2n}$</td>
<td>\ldots</td>
<td>$\lim_{n \to \infty} A_0\left(1 + \frac{r}{n}\right)^{nt}$</td>
</tr>
<tr>
<td>(compounded continuously)</td>
<td>$= A_0$</td>
<td>$= A_0 e^r$</td>
<td>$= A_0 e^{2r}$</td>
<td>\ldots</td>
<td>$= A_0 e^{rt}$</td>
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</tbody>
</table>
Example  If I borrow $50,000 at a 10% interest rate for 5 years with the interest compounded quarterly, how much will I owe after 5 years?
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Example  If I borrow $50,000 at a 10% interest rate for 5 years with the interest compounded quarterly, how much will I owe after 5 years?

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\[ A(t) = 50,000 \left(1 + \frac{1}{4}\right)^{4t} \]
Compound Interest

**Example** If I borrow $50,000 at a 10% interest rate for 5 years with the interest compounded quarterly, how much will I owe after 5 years?

- \( A(t) = A_0(1 + \frac{r}{n})^{nt} \)
- \( A(t) = 50,000(1 + \frac{1}{4})^{4t} \)
- \( A(5) = 50,000(1 + \frac{1}{4})^{20} \approx 81,930.82 \)
The trigonometric function \( \sin x \) is not one-to-one functions, hence in order to create an inverse, we must restrict its domain. The restricted sine function is given by

\[
 f(x) = \begin{cases} 
 \sin x & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
 \text{undefined} & \text{otherwise}
\end{cases}
\]

We have Domain\((f) = [-\frac{\pi}{2}, \frac{\pi}{2}]\) and Range\((f) = [-1, 1]\).
Inverse Sine Function \((\arcsin x = \sin^{-1} x)\).

We see from the graph of the restricted sine function (or from its derivative) that the function is one-to-one and hence has an inverse, shown in red in the diagram below.

This inverse function, \(f^{-1}(x)\), is denoted by \(f^{-1}(x) = \sin^{-1} x \text{ or } \arcsin x\).
Properties of $\sin^{-1} x$.

Domain($\sin^{-1}$) = $[-1, 1]$ and Range($\sin^{-1}$) = $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Since $f^{-1}(x) = y$ if and only if $f(y) = x$, we have:

$\sin^{-1} x = y$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Since $f(f^{-1})(x) = x$ $f^{-1}(f(x)) = x$ we have:

$\sin(\sin^{-1}(x)) = x$ for $x \in [-1, 1]$ $\sin^{-1}(\sin(x)) = x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

From the graph: $\sin^{-1} x$ is an odd function and $\sin^{-1}(-x) = -\sin^{-1} x$. 
Evaluating $\sin^{-1} x$.

**Example** Evaluate $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ using the graph above.

**Example** Evaluate $\sin^{-1}(\sqrt{3}/2)$ and $\sin^{-1}(-\sqrt{3}/2)$.
Evaluating $\sin^{-1} x$.

**Example** Evaluate $\sin^{-1} \left( \frac{-1}{\sqrt{2}} \right)$ using the graph above.

- We see that the point $\left( \frac{-1}{\sqrt{2}}, \frac{\pi}{4} \right)$ is on the graph of $y = \sin^{-1} x$.

**Example** Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. 

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Evaluating $\sin^{-1} x$.

**Example** Evaluate $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ using the graph above.

- We see that the point $\left(\frac{-1}{\sqrt{2}}, \frac{\pi}{4}\right)$ is on the graph of $y = \sin^{-1} x$.
- Therefore $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

**Example** Evaluate $\sin^{-1}(\sqrt{3}/2)$ and $\sin^{-1}(-\sqrt{3}/2)$. 

Annette Pilkington

Exponential Growth and Inverse Trigonometric Functions
Evaluating $\sin^{-1} x$.

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- We see that the point $\left(\frac{-1}{\sqrt{2}}, \frac{\pi}{4}\right)$ is on the graph of $y = \sin^{-1} x$.
- Therefore $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

**Example** Evaluate $\sin^{-1}(\sqrt{3}/2)$ and $\sin^{-1}(-\sqrt{3}/2)$.

- $\sin^{-1}(\sqrt{3}/2) = y$ is the same statement as: $y$ is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin y = \sqrt{3}/2$.  
- $\sin^{-1}(-\sqrt{3}/2) = \text{same as the previous example but negative}$.
Evaluating $\sin^{-1} x$.

**Example** Evaluate $\sin^{-1} \left( \frac{-1}{\sqrt{2}} \right)$ using the graph above.

- We see that the point $\left( \frac{-1}{\sqrt{2}}, \frac{\pi}{4} \right)$ is on the graph of $y = \sin^{-1} x$.
- Therefore $\sin^{-1} \left( \frac{-1}{\sqrt{2}} \right) = \frac{\pi}{4}$.

**Example** Evaluate $\sin^{-1} (\sqrt{3}/2)$ and $\sin^{-1} (-\sqrt{3}/2)$.

- $\sin^{-1} (\sqrt{3}/2) = y$ is the same statement as:
  y is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin y = \sqrt{3}/2$.
- Consulting our unit circle, we see that $y = \frac{\pi}{3}$.
Evaluating $\sin^{-1} x$.

**Example** Evaluate $\sin^{-1} \left( \frac{-1}{\sqrt{2}} \right)$ using the graph above.

- We see that the point $\left( \frac{-1}{\sqrt{2}}, \frac{\pi}{4} \right)$ is on the graph of $y = \sin^{-1} x$.
- Therefore $\sin^{-1} \left( \frac{-1}{\sqrt{2}} \right) = \frac{\pi}{4}$.

**Example** Evaluate $\sin^{-1} (\sqrt{3}/2)$ and $\sin^{-1} (-\sqrt{3}/2)$.

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  $y$ is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin y = \sqrt{3}/2$.
- Consulting our unit circle, we see that $y = \frac{\pi}{3}$.

- $\sin^{-1} (-\sqrt{3}/2) = -\sin^{-1} (\sqrt{3}/2) = -\frac{\pi}{3}$.
More Examples For $\sin^{-1} x$

**Example** Evaluate $\sin^{-1}(\sin \pi)$.

**Example** Evaluate $\cos(\sin^{-1}(\sqrt{3}/2))$. 
More Examples For $\sin^{-1} x$

Example Evaluate $\sin^{-1}(\sin \pi)$.

- We have $\sin \pi = 0$, hence $\sin^{-1}(\sin \pi) = \sin^{-1}(0) = 0$.

Example Evaluate $\cos(\sin^{-1}(\sqrt{3}/2))$.
More Examples For $\sin^{-1} x$

**Example** Evaluate $\sin^{-1}(\sin \pi)$.

- *We have* $\sin \pi = 0$, *hence* $\sin^{-1}(\sin \pi) = \sin^{-1}(0) = 0$.

**Example** Evaluate $\cos(\sin^{-1}(\sqrt{3}/2))$.

- *We saw above that* $\sin^{-1}(\sqrt{3}/2) = \frac{\pi}{3}$. 
More Examples For $\sin^{-1} x$

**Example** Evaluate $\sin^{-1}(\sin \pi)$.

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**Example** Evaluate $\cos(\sin^{-1}(\sqrt{3}/2))$.

- *We saw above that* $\sin^{-1}(\sqrt{3}/2) = \frac{\pi}{3}$.

- *Therefore* $\cos(\sin^{-1}(\sqrt{3}/2)) = \cos\left(\frac{\pi}{3}\right) = 1/2$. 
Example Give a formula in terms of $x$ for $\tan(\sin^{-1}(x))$
Example  Give a formula in terms of \( x \) for \( \tan(\sin^{-1}(x)) \)

- We draw a right angled triangle with \( \theta = \sin^{-1} x \).
Example  Give a formula in terms of $x$ for $\tan(\sin^{-1}(x))$

- We draw a right angled triangle with $\theta = \sin^{-1} x$.

- From this we see that $\tan(\sin^{-1}(x)) = \tan(\theta) = \frac{x}{\sqrt{1-x^2}}$. 

\[
\begin{align*}
\theta & \\
\sqrt{1-x^2} & \\
x & \\
1 & \\
\end{align*}
\]
Derivative of $\sin^{-1} x$.

\[
\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad -1 \leq x \leq 1.
\]

Please read through the proof given in your notes using implicit differentiation. We can also derive a formula for \( \frac{d}{dx} \sin^{-1}(k(x)) \) using the chain rule, or we can apply the above formula along with the chain rule directly.

**Example** Find the derivative

\[
\frac{d}{dx} \sin^{-1} \sqrt{\cos x}
\]
Derivative of $\sin^{-1} x$.

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\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \quad -1 \leq x \leq 1.
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**Example**
Find the derivative

\[
\frac{d}{dx} \sin^{-1} \sqrt{\cos x}
\]

▶ We have $\frac{d}{dx} \sin^{-1} \sqrt{\cos x} = \frac{1}{\sqrt{1-(\sqrt{\cos x})^2}} \frac{d}{dx} \sqrt{\cos x}$.
Derivative of $\sin^{-1} x$.

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad -1 \leq x \leq 1.$$ 

Please read through the proof given in your notes using implicit differentiation. We can also derive a formula for $\frac{d}{dx} \sin^{-1}(k(x))$ using the chain rule, or we can apply the above formula along with the chain rule directly.

**Example** Find the derivative

$$\frac{d}{dx} \sin^{-1} \sqrt{\cos x}.$$ 

- We have $\frac{d}{dx} \sin^{-1} \sqrt{\cos x} = \frac{1}{\sqrt{1-(\sqrt{\cos x})^2}} \frac{d}{dx} \sqrt{\cos x}$

- $$= \frac{1}{\sqrt{1-\cos x}} \cdot \frac{-\sin x}{2\sqrt{\cos x}} = \frac{-\sin x}{2\sqrt{\cos x}\sqrt{1-\cos x}}$$
We can define the function $\cos^{-1} x$ similarly. You can read the definition in your book. It can be shown that $\frac{d}{dx} \cos^{-1} x = -\frac{d}{dx} \sin^{-1} x$ and one can use this to prove that

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$
The tangent function is not a one to one function. The **restricted tangent function** is given by

\[
h(x) = \begin{cases} 
  \tan x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\
  \text{undefined} & \text{otherwise}
\end{cases}
\]

We see from the graph of the restricted tangent function (or from its derivative) that the function is one-to-one and hence has an inverse, which we denote by

\[
h^{-1}(x) = \tan^{-1} x \text{ or } \arctan x.
\]
Graphs of Restricted Tangent and $\tan^{-1}x$. 

\[ y = h(x) \]

\[ y = \arctan(x) \]
Properties of $\tan^{-1}x$.

Domain($\tan^{-1}$) = $(-\infty, \infty)$ and Range($\tan^{-1}$) = $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Since $h^{-1}(x) = y$ if and only if $h(y) = x$, we have:

$\tan^{-1}x = y$ if and only if $\tan(y) = x$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Since $h(h^{-1}(x)) = x$ and $h^{-1}(h(x)) = x$, we have:

$\tan(\tan^{-1}(x)) = x$ for $x \in (-\infty, \infty)$ and $\tan^{-1}(\tan(x)) = x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

From the graph, we have: $\tan^{-1}(-x) = -\tan^{-1}(x)$.

Also, since $\lim_{x \to \left(\frac{\pi}{2}^{-}\right)} \tan x = \infty$ and $\lim_{x \to \left(-\frac{\pi}{2}^{+}\right)} \tan x = -\infty$,

we have $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$ and $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$.
Evaluating $\tan^{-1} x$

**Example** Find $\tan^{-1}(1)$ and $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

**Example** Find $\cos(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right))$. 

![Diagram of the unit circle with angles and sine and cosine values.](diagram.png)
Evaluating $\tan^{-1} x$

**Example** Find $\tan^{-1}(1)$ and $\tan^{-1}(\frac{1}{\sqrt{3}})$.

- $\tan^{-1}(1)$ is the unique angle, $\theta$, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan \theta = \frac{\sin \theta}{\cos \theta} = 1$. By inspecting the unit circle, we see that $\theta = \frac{\pi}{4}$.

**Example** Find $\cos(\tan^{-1}(\frac{1}{\sqrt{3}}))$. 

![Unit Circle Diagram](image)
Evaluating $\tan^{-1} x$

**Example** Find $\tan^{-1}(1)$ and $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

- $\tan^{-1}(1)$ is the unique angle, $\theta$, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan \theta = \frac{\sin \theta}{\cos \theta} = 1$. By inspecting the unit circle, we see that $\theta = \frac{\pi}{4}$.

- $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is the unique angle, $\theta$, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{1}{\sqrt{3}}$. By inspecting the unit circle, we see that $\theta = \frac{\pi}{6}$.

**Example** Find $\cos(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right))$. 

$\cos(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$. 

---

**Figure Description:**

- The unit circle is divided into various sections, each labeled with angles in radians and degrees.
- Key points are marked with their corresponding coordinates and angles, including $(0, 1)$, $(1, 0)$, $(0, -1)$, and $(1, 1)$.
Evaluating $\tan^{-1} x$

**Example** Find $\tan^{-1}(1)$ and $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

- $\tan^{-1}(1)$ is the unique angle, $\theta$, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan \theta = \frac{\sin \theta}{\cos \theta} = 1$. By inspecting the unit circle, we see that $\theta = \frac{\pi}{4}$.

- $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is the unique angle, $\theta$, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{1}{\sqrt{3}}$. By inspecting the unit circle, we see that $\theta = \frac{\pi}{6}$.

**Example** Find $\cos(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right))$.

- $\cos(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.
Derivative of $\tan^{-1} x$.

Using implicit differentiation, we get

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}, \quad -\infty < x < \infty.$$ 

(Please read through the proof in your notes.) We can use the chain rule in conjunction with the above derivative.

**Example** Find the domain and derivative of $\tan^{-1}(\ln x)$
Derivative of $\tan^{-1} x$.

Using implicit differentiation, we get

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}, \quad -\infty < x < \infty.$$ 

(Please read through the proof in your notes.) We can use the chain rule in conjunction with the above derivative.

**Example** Find the domain and derivative of $\tan^{-1}(\ln x)$

- Domain = Domain of $\ln x = (0, \infty)$
Using implicit differentiation, we get

\[ \frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}, \quad -\infty < x < \infty. \]

(Please read through the proof in your notes.) We can use the chain rule in conjunction with the above derivative.

Example Find the domain and derivative of \( \tan^{-1}(\ln x) \)

\[ \text{Domain} = \text{Domain of } \ln x = (0, \infty) \]

\[ \frac{d}{dx} \tan^{-1}(\ln x) = \frac{1}{x} \cdot \frac{1}{1 + (\ln x)^2} = \frac{1}{x(1 + (\ln x)^2)}. \]
Integration Formulas

Reversing the derivative formulas above, we get

\[ \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C, \quad \int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + C, \]

Example

\[ \int_{0}^{1/2} \frac{1}{1 + 4x^2} \, dx \]
Integration Formulas

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**Example**

\[ \int_{0}^{1/2} \frac{1}{1 + 4x^2} \, dx \]

- **We use substitution.** Let \( u = 2x \), then \( du = 2 \, dx \), \( u(0) = 0 \), \( u(1/2) = 1 \).
Integration Formulas

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Example

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\int_{0}^{1/2} \frac{1}{1 + 4x^2} \, dx
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We use substitution. Let \( u = 2x \), then \( du = 2 \, dx \), \( u(0) = 0 \), \( u(1/2) = 1 \).

\[
\int_{0}^{1/2} \frac{1}{1 + 4x^2} \, dx = \frac{1}{2} \int_{0}^{1} \frac{1}{1 + u^2} \, du = \frac{1}{2} \tan^{-1} u|_{0}^{1} = \frac{1}{2} [\tan^{-1}(1) - \tan^{-1}(0)]
\]
Reversing the derivative formulas above, we get

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C, \quad \int \frac{1}{x^2 + 1} \, dx = \tan^{-1} x + C,
\]

**Example**

\[
\int_0^{1/2} \frac{1}{1 + 4x^2} \, dx
\]

We use substitution. Let \( u = 2x \), then \( du = 2\,dx \), \( u(0) = 0 \), \( u(1/2) = 1 \).

\[
\int_0^{1/2} \frac{1}{1 + 4x^2} \, dx = \frac{1}{2} \int_0^1 \frac{1}{1 + u^2} \, du = \frac{1}{2} \tan^{-1} u|_0^1 = \frac{1}{2} [\tan^{-1}(1) - \tan^{-1}(0)]
\]

\[
= \frac{1}{2} \left[ \frac{\pi}{4} - 0 \right] = \frac{\pi}{8}.
\]
Exponential growth and Decay
Inverse Trigonometric functions. Inverse Tangent Function
Indeterminate Forms
Integration

Example

\[ \int \frac{1}{\sqrt{9 - x^2}} \, dx \]
Exponential growth and Decay  Inverse Trigonometric functions  Inverse Tangent Function Indeterminate Forms

Integration

Example

\[
\int \frac{1}{\sqrt{9 - x^2}} \, dx
\]

\[
\int \frac{1}{\sqrt{9 - x^2}} \, dx = \int \frac{1}{3\sqrt{1 - \frac{x^2}{9}}} \, dx = \frac{1}{3} \int \frac{1}{\sqrt{1 - \frac{x^2}{9}}} \, dx
\]
Example

\[
\int \frac{1}{\sqrt{9 - x^2}} \, dx
\]

\[
\int \frac{1}{\sqrt{9 - x^2}} \, dx = \int \frac{1}{3 \sqrt{1 - \frac{x^2}{9}}} \, dx = \frac{1}{3} \int \frac{1}{\sqrt{1 - \frac{x^2}{9}}} \, dx
\]

Let \( u = \frac{x}{3} \), then \( dx = 3 \, du \)
Integration

Example

\[ \int \frac{1}{\sqrt{9 - x^2}} \, dx \]

\[ \int \frac{1}{\sqrt{9 - x^2}} \, dx = \int \frac{1}{3\sqrt{1 - \frac{x^2}{9}}} \, dx = \frac{1}{3} \int \frac{1}{\sqrt{1 - \frac{x^2}{9}}} \, dx \]

\[ \text{Let } u = \frac{x}{3}, \text{ then } dx = 3du \]

\[ \int \frac{1}{\sqrt{9 - x^2}} \, dx = \frac{1}{3} \int \frac{3}{\sqrt{1 - u^2}} \, du = \sin^{-1} u + C = \sin^{-1} \left( \frac{x}{3} \right) + C \]
Indeterminate forms of type $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

**Definition** An indeterminate form of the type $\frac{0}{0}$ is a limit of a quotient where both numerator and denominator approach 0.

**Example**

\[
\lim_{x \to 0} \frac{e^x - 1}{\sin x}, \quad \lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}, \quad \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}
\]

**Definition** An indeterminate form of the type $\frac{\infty}{\infty}$ is a limit of a quotient $\frac{f(x)}{g(x)}$ where $f(x) \to \infty$ or $-\infty$ and $g(x) \to \infty$ or $-\infty$.

**Example**

\[
\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x}, \quad \lim_{x \to 0^+} \frac{x^{-1}}{\ln x}.
\]
Indeterminate forms of type \( \frac{0}{0} \) and \( \frac{\infty}{\infty} \).

**L’Hospital’s Rule**    Suppose \( \lim \) stands for any one of

\[
\lim_{x \to a} \quad \lim_{x \to a^+} \quad \lim_{x \to a^-} \quad \lim_{x \to \infty} \quad \lim_{x \to -\infty}
\]

and \( \frac{f(x)}{g(x)} \) is an indeterminate form of type \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).

If \( \lim \frac{f'(x)}{g'(x)} \) is a finite number \( L \) or is \( \pm \infty \), then

\[
\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.
\]

(Assuming that \( f(x) \) and \( g(x) \) are both differentiable in some open interval around \( a \) or \( \infty \) (as appropriate) except possible at \( a \), and that \( g'(x) \neq 0 \) in that interval).
Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x}$$
Examples of Indeterminate forms of type $\frac{0}{0}$.

**Example** Find

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x}$$

- *Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L’Hospital’s rule.*
Examples of Indeterminate forms of type \( \frac{0}{0} \).

**Example** Find

\[
\lim_{x \to 0} \frac{e^x - 1}{\sin x}
\]

- Since this is an indeterminate form of type \( \frac{0}{0} \), we can apply L’Hospital’s rule.

\[
\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \left( L’Hosp. \right) \lim_{x \to 0} \frac{e^x}{\cos x} = \left( Eval. \right) 1
\]
Examples of Indeterminate forms of type $\frac{0}{0}$.

**Example** Find

$$\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}$$
Examples of Indeterminate forms of type $\frac{0}{0}$.

Example Find

$$\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}$$

Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L'Hospital’s rule.
Examples of Indeterminate forms of type $\frac{0}{0}$.

**Example** Find

$$\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}$$

- Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L'Hospital's rule.

- As it stands, this quotient gets more complicated when we apply L'Hospital's rule, so we rearrange it before we apply the rule.
Examples of Indeterminate forms of type $\frac{0}{0}$.

**Example** Find

$$\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}$$

- Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L'Hospital’s rule.

- As it stands, this quotient gets more complicated when we apply L'Hospital’s rule, so we rearrange it before we apply the rule.

$$\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}} = \lim_{x \to \infty} \frac{1/x^2}{e^{-x}} = \lim_{x \to \infty} \frac{e^x}{x^2}$$
Examples of Indeterminate forms of type $\frac{0}{0}$.

**Example** Find

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$$\lim_{x \to \infty} \frac{e^x}{x^2} = (L'Hosp.) \lim_{x \to \infty} \frac{e^x}{2x} = (L'Hosp.) \lim_{x \to \infty} \frac{e^x}{2}$$
Examples of Indeterminate forms of type $0/0$.

**Example** Find

\[
\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}}
\]

- Since this is an indeterminate form of type $0/0$, we can apply L'Hospital’s rule.

- As it stands, this quotient gets more complicated when we apply L'Hospital’s rule, so we rearrange it before we apply the rule.

\[
\lim_{x \to \infty} \frac{x^{-2}}{e^{-x}} = \lim_{x \to \infty} \frac{1/x^2}{e^{-x}} = \lim_{x \to \infty} \frac{e^x}{x^2}
\]

- \[
\lim_{x \to \infty} \frac{e^x}{x^2} = (L' Hosp.) \lim_{x \to \infty} \frac{e^x}{2x} = (L' Hosp.) \lim_{x \to \infty} \frac{e^x}{2}
\]

- As $x \to \infty$, we have $e^x \to \infty$ and therefore $\lim_{x \to \infty} \frac{e^x}{2} = \infty$. 
Examples of Indeterminate forms of type $\frac{0}{0}$.

**Example** Find

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$
Examples of Indeterminate forms of type $\frac{0}{0}$.

**Example** Find

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$$

- **Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L’Hospital’s rule.** (Cos $x$ and $x - \frac{\pi}{2}$ are both differentiable everywhere and $g'(x) \neq 0$ where $g(x) = x - \frac{\pi}{2}$).
Examples of Indeterminate forms of type $\frac{0}{0}$.

**Example** Find

$$\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2}$$

- Since this is an indeterminate form of type $\frac{0}{0}$, we can apply L’Hospital’s rule. ( $\cos x$ and $x - \pi/2$ are both differentiable everywhere and $g'(x) \neq 0$ where $g(x) = x - \pi/2$).

$$\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2} = \lim_{x \to \pi/2} \frac{-\sin x}{1} = -1 \quad \text{(Eval.)}$$
Example Find

\[ \lim_{{x \to \infty}} \frac{{x^2 + 2x + 1}}{e^x} \]
Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

**Example** Find

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x}$$

- *Since this is an indeterminate form of type $\frac{\infty}{\infty}$, we can apply L'Hospital's rule.*
Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

**Example** Find

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x}$$

Since this is an indeterminate form of type $\frac{\infty}{\infty}$, we can apply L'Hospital's rule.

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x} = (L'Hosp.) \lim_{x \to \infty} \frac{2x + 2}{e^x} = (L'Hosp.) \lim_{x \to \infty} \frac{2}{e^x} = 0$$
Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

**Example** Find

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x}$$

- Since this is an indeterminate form of type $\frac{\infty}{\infty}$, we can apply L'Hospital's rule.

$$\lim_{x \to \infty} \frac{x^2 + 2x + 1}{e^x} = \lim_{x \to \infty} \frac{2x + 2}{e^x} = \lim_{x \to \infty} \frac{2}{e^x}$$

- As $x \to \infty$, we have $e^x \to \infty$ and therefore $\lim_{x \to \infty} \frac{2}{e^x} = 0$. 
Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

**Example** Find

$$\lim_{x \to 0^+} \frac{x^{-1}}{\ln(x)}$$
Examples of Indeterminate forms of type \( \frac{\infty}{\infty} \).

Example Find

\[
\lim_{{x \to 0^+}} \frac{x^{-1}}{\ln(x)}
\]

- Since this is an indeterminate form of type \( \frac{\infty}{\infty} \), we can apply L'Hospital's rule.
Examples of Indeterminate forms of type $\frac{\infty}{\infty}$.

**Example** Find

$$\lim_{x \to 0^+} \frac{x^{-1}}{\ln(x)}$$

**Since this is an indeterminate form of type $\frac{\infty}{\infty}$, we can apply L'Hospital's rule.**

$$\lim_{x \to 0^+} \frac{x^{-1}}{\ln(x)} = \lim_{x \to 0^+} -\frac{x^{-2}}{1/x} = \lim_{x \to 0^+} -\frac{1/x^2}{1/x} = \lim_{x \to 0^+} -\frac{1}{x} = -\infty$$
Indeterminate forms of type $0 \cdot \infty$.

**Definition** \( \lim f(x)g(x) \) is an indeterminate form of the type $0 \cdot \infty$ if \[
\lim f(x) = 0 \quad \text{and} \quad \lim g(x) = \pm\infty.
\]

**Example** \( \lim_{x \to 0} x \ln |x| \)

We can convert the above indeterminate form to an indeterminate form of type $0 \cdot 0$ by writing \[
f(x)g(x) = \frac{f(x)}{1/g(x)}
\]
or to an indeterminate form of the type $\infty/\infty$ by writing \[
f(x)g(x) = \frac{g(x)}{1/f(x)}.
\]

We then apply L’Hospital’s rule to the limit.
Example of an Indeterminate form of type $0 \cdot \infty$.

Example: $\lim_{x \to 0} x \ln |x|$
Example of an Indeterminate form of type $0 \cdot \infty$.

**Example** $\lim_{x \to 0} x \ln |x|$

- We can convert the above indeterminate form to an indeterminate form of type $\frac{\infty}{\infty}$ by writing

$$f(x)g(x) = \frac{g(x)}{1/f(x)}.$$
Example of an Indeterminate form of type $0 \cdot \infty$. 

**Example** $\lim_{x \to 0} x \ln |x|$

- We can convert the above indeterminate form to an indeterminate form of type $\frac{\infty}{\infty}$ by writing

$$f(x)g(x) = \frac{g(x)}{1/f(x)}.$$

- $\lim_{x \to 0} x \ln |x| = \lim_{x \to 0} \frac{\ln |x|}{1/x}$. 

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Exponential Growth and Inverse Trigonometric Functions
Example of an Indeterminate form of type $0 \cdot \infty$.

**Example** \( \lim_{x \to 0} x \ln |x| \)

- We can convert the above indeterminate form to an indeterminate form of type $\infty / \infty$ by writing
  \[ f(x)g(x) = \frac{g(x)}{1/f(x)}. \]

- \( \lim_{x \to 0} x \ln |x| = \lim_{x \to 0} \frac{\ln |x|}{1/x} \).

- We then apply L’Hospital’s rule to the limit.
Example of an Indeterminate form of type $0 \cdot \infty$.

**Example** $\lim_{x \to 0} x \ln |x|$

- We can convert the above indeterminate form to an indeterminate form of type $\infty \cdot \infty$ by writing
  \[ f(x)g(x) = \frac{g(x)}{1/f(x)}. \]

- $\lim_{x \to 0} x \ln |x| = \lim_{x \to 0} \frac{\ln |x|}{1/x}$.

- We then apply L’Hospital’s rule to the limit.

- $\lim_{x \to 0} \frac{\ln |x|}{1/x} = \lim_{x \to 0} \frac{1/x}{(-1/x^2)} = \lim_{x \to 0} \frac{1}{(-1/x)} = \lim_{x \to 0} (-x) = 0$
Indeterminate forms of type $0^0$, $\infty^0$, $1^\infty$. 

<table>
<thead>
<tr>
<th>Type</th>
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<th>$\lim g(x) = 0$</th>
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**Example**  
$\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$.

**Method**
Indeterminate forms of type $0^0$, $\infty^0$, $1^\infty$.

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Example $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$.

Method

- Look at $\lim \ln[f(x)]^{g(x)} = \lim g(x) \ln[f(x)]$. 
Indeterminate forms of type $0^0$, $\infty^0$, $1^\infty$.

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| $\infty^0$ | $\lim [f(x)]^{g(x)}$  
$\lim f(x) = \infty$  
$\lim g(x) = 0$ |
| $1^\infty$ | $\lim [f(x)]^{g(x)}$  
$\lim f(x) = 1$  
$\lim g(x) = \infty$ |

**Example**  
$\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$.

**Method**

- Look at $\lim \ln [f(x)]^{g(x)} = \lim g(x) \ln [f(x)]$.
- Use L’Hospital to find $\lim g(x) \ln [f(x)] = \alpha$. ($\alpha$ might be finite or $\pm \infty$ here.)
Indeterminate forms of type $0^0$, $\infty^0$, $1^\infty$.

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Example  \( \lim_{x \to 0} (1 + x)^{\frac{1}{x}} \).

Method

- Look at $\lim \ln[f(x)]^{g(x)} = \lim g(x) \ln[f(x)]$.
- Use L’Hospital to find $\lim g(x) \ln[f(x)] = \alpha$. ($\alpha$ might be finite or $\pm \infty$ here.)
- Then $\lim f(x)^{g(x)} = \lim e^{\ln[f(x)]^{g(x)}} = e^\alpha$ since $e^x$ is a continuous function. (where $e^{\infty}$ should be interpreted as $\infty$ and $e^{-\infty}$ should be interpreted as 0.)
Example of an Indeterminate form of type $1^\infty$.

Example $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$.

Method

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Example of an Indeterminate form of type $1^\infty$.

Example  \( \lim_{x \to 0} (1 + x)^{\frac{1}{x}} \).

Method

- Look at \( \lim \ln[f(x)]^{g(x)} = \lim g(x) \ln[f(x)] \):
  - Look at \( \lim_{x \to 0} \ln[1 + x] \)
  - \( \lim_{x \to 0} \frac{1}{x} \ln[1 + x] \)

Examples: 

- \( \lim_{x \to 0} (1 + x)^{\frac{1}{x}} \)
Example of an Indeterminate form of type $1^\infty$.

**Example**  \( \lim_{x \to 0} (1 + x)^{1/x} \).

**Method**

- **Look at** \( \lim \ln[f(x)]^{g(x)} = \lim g(x) \ln[f(x)] \): **Look at**
  \( \lim_{x \to 0} \ln[1 + x]^{1/x} = \lim_{x \to 0} \frac{1}{x} \ln[1 + x] \)

- **Use L’Hospital to find** \( \lim g(x) \ln[f(x)] = \alpha \).

\[
\lim_{x \to 0} \frac{1}{x} \ln[1 + x] = \lim_{x \to 0} \frac{\ln[1 + x]}{x} = \alpha \\
\lim_{x \to 0} \frac{1/[1 + x]}{1} = 1 (= \alpha).
\]
Example of an Indeterminate form of type $1^{\infty}$.

**Example**  \( \lim_{x \to 0} (1 + x)^{\frac{1}{x}} \).

**Method**

- *Look at* \( \lim \ln[f(x)]^{g(x)} = \lim g(x) \ln[f(x)] : \) *Look at* \( \lim_{x \to 0} \ln[1 + x]^{\frac{1}{x}} = \lim_{x \to 0} \frac{1}{x} \ln[1 + x] \)

- *Use L’Hospital to find* \( \lim g(x) \ln[f(x)] = \alpha \).

\[
\lim_{x \to 0} \frac{1}{x} \ln[1 + x] = \lim_{x \to 0} \frac{\ln[1 + x]}{x} = \lim_{x \to 0} \frac{1/[1 + x]}{1} = 1 (= \alpha).
\]  

(L’Hosp.)

- *Then* \( \lim f(x)^{g(x)} = \lim e^{\ln[f(x)]^{g(x)}} = e^{\lim \ln[f(x)]^{g(x)}} = e^\alpha \)

\[
\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = \lim_{x \to 0} e^{\ln[(1+x)^{\frac{1}{x}}]} = e^{\lim_{x \to 0} \ln[(1+x)^{\frac{1}{x}}]} = e^1 = e.
\]
Indeterminate forms of type $\infty - \infty$.

Indeterminate Forms of the type $\infty - \infty$ occur when we encounter a limit of the form $\lim(f(x) - g(x))$ where $\lim f(x) = \lim g(x) = \infty$ or $\lim f(x) = \lim g(x) = -\infty$.

To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...

Example $\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x}$.
Indeterminate forms of type $\infty - \infty$.

**Indeterminate Forms of the type** $\infty - \infty$ occur when we encounter a limit of the form

$$\lim(f(x) - g(x))$$

where $\lim f(x) = \lim g(x) = \infty$ or $\lim f(x) = \lim g(x) = -\infty$

To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...

**Example**

$$\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x}$$

$$\Rightarrow \lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x}$$
Indeterminate forms of type $\infty - \infty$.

**Indeterminate Forms of the type** $\infty - \infty$ occur when we encounter a limit of the form

$$\lim (f(x) - g(x))$$

where $\lim f(x) = \lim g(x) = \infty$ or

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To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...

**Example**  
$$\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x}$$

- $$\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x}$$

- *This is an indeterminate form of type $0/0$ so we can use L'Hospital.*
Indeterminate forms of type $\infty - \infty$.

**Indeterminate Forms of the type** $\infty - \infty$ occur when we encounter a limit of the form

$$\lim_{x \to a} (f(x) - g(x))$$

where $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$ or $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = -\infty$

To deal with these limits, we try to convert to the previous indeterminate forms by adding fractions etc...

**Example**

$$\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x}$$

$$= \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \quad (\text{L' Hospital})$$

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$$

$$= \lim_{x \to 0^+} \frac{-\sin x}{\cos x + (\cos x - x \sin x)} = 0$$

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