Trigonometric Integrals you can already figure out.

Trigonometric Formulas : Essential Background

$$\sin^2 x + \cos^2 x = 1$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\sec x = \sec x \tan x,$$

Use the following identities

$$\sin((m-n)x) = \sin(mx)\cos(nx) - \cos(mx)\sin(nx)$$

$$\sin((m+n)x) = \sin mx \cos nx + \sin nx \cos mx$$

$$\cos((m-n)x) = \cos(mx)\cos(nx) + \sin(nx)\sin(mx)$$

$$\cos((m+n)x) = \cos(mx)\cos(nx) - \sin(nx)\sin(mx)$$

to get

$$\sin(mx)\cos(nx) = \frac{1}{2}\Big[\sin((m-n)x) + \sin((m+n)x\Big]$$

$$\sin(mx)\sin(nx) = \frac{1}{2}\Big[\cos((m-n)x) - \cos((m+n)x\Big]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\Big[\cos((m-n)x) + \cos((m+n)x\Big]$$

Integrating powers of trigonometric functions

Small Powers of Sine

$$\int \sin^0 x \; dx = \int 1 \; dx = x + C.$$

$$\int \sin^1 x \; dx = -\cos x + C.$$

$$\int \sin^2 x \; dx = \int \frac{1}{2}(1 - \cos 2x) \; dx = \frac{1}{2}[x - \frac{\sin 2x}{2}] + C.$$

$$\int \sin^3 x \; dx = \int \sin^2 x \sin x \; dx = \int [1 - \cos^2 x] \sin x \; dx$$

$$\text{Let} u = \cos x, \quad du = -\sin x \; dx, \quad \int \sin^3 x \; dx = -\int [1 - u^2] \; du = -[u - \frac{u^3}{3}] + C$$

$$= -[\cos x - \frac{\cos^3 x}{3}] + C = \frac{\cos^3 x}{3} - \cos x + C.$$

Small Powers of Cosine

$$\int \cos^0 x \; dx = \int 1 \; dx = x + C.$$

$$\int \cos^1 x \, dx = -\sin x + C.$$

$$\int \cos^2 x \, dx = \int \frac{1}{2}(1+\cos 2x) \, dx = \frac{1}{2}[x + \frac{\sin 2x}{2}] + C.$$

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int [1-\sin^2 x] \cos x \, dx$$

$$\text{Let} u = \sin x, \quad du = \cos x \, dx, \quad \int \cos^3 x \, dx = \int [1-u^2] \, du = [u - \frac{u^3}{3}] + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C.$$

Small Powers of tangent

$$\int \tan^0 x dx = \int 1 dx = x + C.$$

$$\int \tan x dx = \ln|\sec x| + C$$

Proof

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

Using the substitution $u = \cos x$, we get $du = -\sin x$ giving us that the above integral is

$$\int \frac{-1}{u} du = -\ln|u| = \ln|\sec x| + C.$$

Example

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

Example

$$\int \tan^3 x dx = \int (\sec^2 x - 1)\tan x dx = \int (\sec^2 x)\tan x dx - \int \tan x dx$$

$$= \frac{\tan^2 x}{2} + \ln|\sec x| + C.$$

Small Powers of Secant

$$\int \sec^0 x dx = \int 1 dx = x + C.$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

Proof

$$\int \sec x dx = \int \sec x \Big(\frac{\sec x + \tan x}{\sec x + \tan x}\Big) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Using the substitution $u = \sec x + \tan x$, we get $du = \sec^2 x + \sec x \tan x$ giving us that the above integral is

$$\int \frac{1}{u} du = \ln |u| = \ln |\sec x + \tan x| + C.$$

Example

$$\int \sec^3 x dx = \int \sec^2 x \sec x \ dx$$

use integration by parts with $u = \sec x, \quad dv = \sec^2 x dx$ to get (a recurring integral)

$$\int \sec^3 x dx = \int \sec^2 x \sec x \ dx = \sec x \tan x - \int \tan^2 x \sec x \ dx = \sec x \tan x - \int (\sec^2 x - 1) \sec x \ dx$$

$$= \sec x \tan x - \int \sec^3 x \ dx + \int \sec x \ dx$$

Solving for $\int \sec^3 x \ dx$, we get

$$\int \sec^3 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec^1 x dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln |\sec x + \tan x| + C.$$