Worksheet

- The purpose of this worksheet is
  1. To understand how the differential equation describing simple harmonic motion is derived.
  2. To explore how to predict what the solution to this differential equation might be using Euler's method and to examine the difficulties of using Euler's method in this case, namely we need a very small step size for the method to work. (There are other methods of finding numerical solutions which are more accurate in this situation which take more computing time. You will learn about these methods in Numerical Analysis.)
  3. To compare the solution derived from Euler's method with the actual solution, which can be derived by some guesswork and a little calculation here.
  4. To compare what happens in reality, when we actually attach a mass to a spring and set it in motion, to our prediction with Euler's method and the theoretical results. Results of the experiment are attached.
- You should download the accompanying `Maple` worksheet `HM.mv`, make a backup copy and check that you can open it using the application `Maple14` or later at one of the labs on campus as soon as possible.
- The solutions are provided in an accompanying document (for an older version of this worksheet).

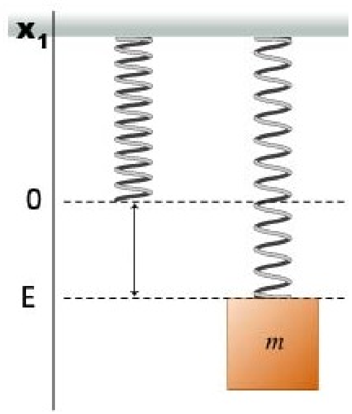
Simple Harmonic Motion

Lets assume we have an object with mass m attached to a spring with spring constant k. Recall that **Hooke's Law** says that the force required to stretch the spring x_1 meters beyond its natural length is kx_1. This means that a spring stretched x_1 meters from its natural length is exerting a force of $-kx_1$ Newtons (using SI units, Newtons, kilograms, meters, seconds.).

Gravity, on the other hand is exerting a downward force on the object, of mg where $g = 9.81$ meters/s^2.

Problem 1 We will use a spring that is stretched 2 cm = 0.02 meters by a force of 1 Newton. What is the spring constant k?
(Recall this is a Calculus 1 problem: from Section 6.4)

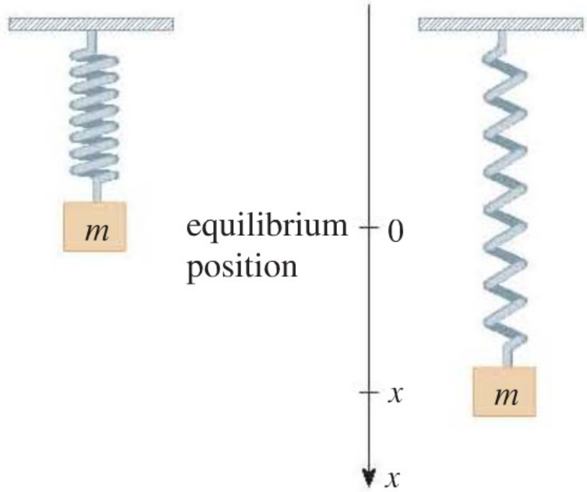
$k =$



Problem 2 If we attach a mass of $m = 100$ g $= 0.1$ kg to the end of the spring discussed in problem 1, what is the equilibrium position for the spring with this weight attached?

Equilibrium Position at $x_1 = E$ (meters) $=$

(Note that $100g$ exerts a force of approx 1N, so the value of E should be close to 2 cm = .02 meters.)



Problem 3 Let us assume that we have attached an object with a mass of 100g = 0.1 kg to the above spring and we stretch the spring to 10 cm = 0.1 meters beyond its equilibrium position.

We let go of the spring, starting the clock as we do so and measuring time, t, in seconds.

Let $x(t)$ be the distance in meters of the object from the equilibrium position at time t.

From experience you probably know that the object will bob up and down. What do you expect the function $x(t)$ to look like?

We would like to restrict our attention to the idealized situation where the ONLY forces acting on the spring are the ones mentioned above, gravity and the spring force. Our question is:
What path will the spring follow in this idealized situation? or What is the function $x(t)$?

Setting up a differential equation to describe the motion

From our definition of $x(t)$ above, we have the **initial condition** $\boxed{x(0) = .1}$ meter.

Let us consider **the forces acting on the spring** when the spring is at x meters from its equilibrium position as shown in the diagram above. The force of gravity is exerting a force of $mg = (.1)9.81N$ in a downward direction (positive direction on the axis). Let E denote the distance the spring is stretched from its natural length by the object in equilibrium position. (This is the number you found in Problem 2). The spring force is acting in the opposite direction to the force of gravity exerting a force of $-k(x+E)N$ on the object. Hence **the total force** on the object, when the object is x meters from its equilibrium position is $\underline{= -k(x+E) + (.1)(9.81)}$ **N.** Now $kE = mg = (.1)(9.81)$ **(Why?)**, hence the total force force on on the object, when the object is x meters from its equilibrium position is $-kx - (.1)(9.81) + (.1)(9.81) \underline{= -kx}$ N.

The sum of the forces acting on the spring at any given value of x are equal to the mass times the acceleration of the spring at that point. Hence for any value of $x = x(t)$:

$$\boxed{ma = -kx} \qquad \text{or} \qquad \boxed{(.1)\frac{d^2x(t)}{dt^2} = -kx(t).}$$

This is a second order differential equation. We do not have a method for finding the solution in this course, however we can apply Euler's method to find a numerical solution.

Using Euler's Method to find a Numerical solution We cannot use Euler's method directly as we did in class to find a solution to this differential equation since it is a second order differential equation. However, we can split the equation into two first order differential equations and apply Euler's method simultaneously to both to find an approximate velocity and position at each step.

Let $v(t) = x'(t)$ denote the velocity of the object at time t

and

let $a(t) = v'(t) = \frac{d^2x(t)}{dt}$ denote the acceleration of the object at time t.

From the differential equation given above, we get the following differential equations:

$$\frac{dv(t)}{dt} = a(t) = \frac{d^2x(t)}{dt} = -10kx(t), \quad \text{or} \quad \boxed{a(t) = \frac{dv(t)}{dt} = -10kx(t) \text{ and } \frac{dx}{dt} = v(t)}$$

Recall that our problem is to predict the path of the object (weighing 100g = 0.1 kg) when we stretch the spring to 10 cm = .1 meters beyond its equilibrium position. This gives us initial values of

$$\boxed{x(0) = 0.1 \quad v(0) = 0} \quad a(0) = -10kx(0) = -10k(.1) = -k.$$

Note you should have found a value for k in problem 1.

We will use Euler's method applied to both equations above simultaneously to make a discrete model predicting the path of the object on the spring over a 5 second period. We will make our predictions using a discrete sequence of points with the time interval changing by $h = \Delta t = .001$ seconds.

At each value of t,

$$t_0 = 0, \quad t_1 = .001, \quad t_2 = .002, \quad \ldots \quad , t_{5000} = 5$$

we will estimate the position, $x(t_i)$, velocity, $v(t_i)$, and acceleration, $a(t_i)$, of the object at time t_i. We will accomplish this using the state equations :

$$\frac{x(t_{i+1}) - x(t_i)}{\Delta t} \approx v(t_i) \quad \text{or} \quad \boxed{x(t_{i+1}) \approx x(t_i) + v(t_i)\Delta t}$$

and

$$\frac{v(t_{i+1}) - v(t_i)}{\Delta t} \approx a(t_i) \approx -10kx(t_i) \quad \text{or} \quad \boxed{v(t_{i+i}) \approx v(t_i) + a(t_i)\Delta t}$$

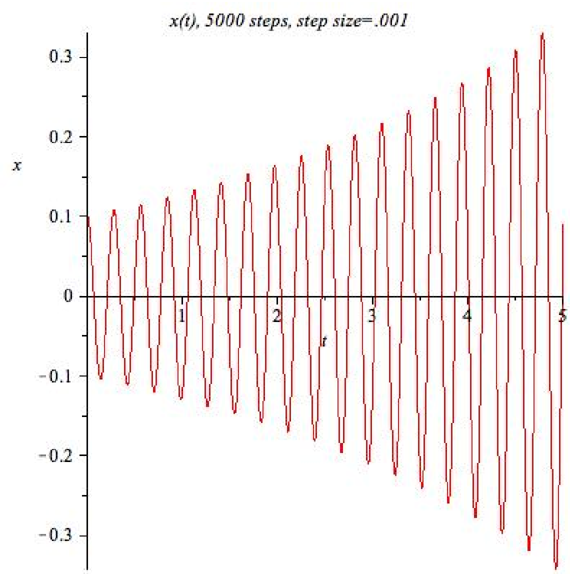
Fill in some values on the table on below, using the value of k that you have calculated in problem 1 above.

t_i	$x(t_i) = x(t_{i-1}) + v(t_{i-1})\Delta t$	$v(t_i) = v(t_{i-1}) + a(t_{i-1})\Delta t$	$a(t_i) = -10kx(t_i)$
$t_0 = 0$	0.1	0	$-k =$
0.001			
0.002			
0.003			
0.004			
0.005			

Using Maple to graph the solution You will find a maple worksheet entitled `HM.mv` accompanying this worksheet. You should download this file and take it to a computer in a computer lab on campus. You can open the file with the application `MAPLE`. You should make a copy of this file before you work through it. You should activate each piece of code (in red) in turn by placing the cursor at the end of a line of code and hitting return. When running the for loop, you can place the cursor at the end of any line of code in that section and hit return. Generate a graph of your numerical solution by activating the last piece of code and print your completed worksheet to turn in with this worksheet.

After you have backed up the file, change the value of h to $h = 0.01$ in the definitions and rerun the worksheet to get a graph of the approximation with the new value of h. You will see the numerical instability in the solution with this new value of h. Print out this completed worksheet and also attach it to you worksheet to turn in.

The graph you generate should look like this:



The Real Solution

You will be able to solve this equation with the methods discussed in the lectures on second order linear differential equations. In fact a similar equation is discussed in the examples. However we can actually guess the solution to this equation.

$$\frac{d^2x(t)}{dt^2} = -10kx(t),$$

with initial conditions:

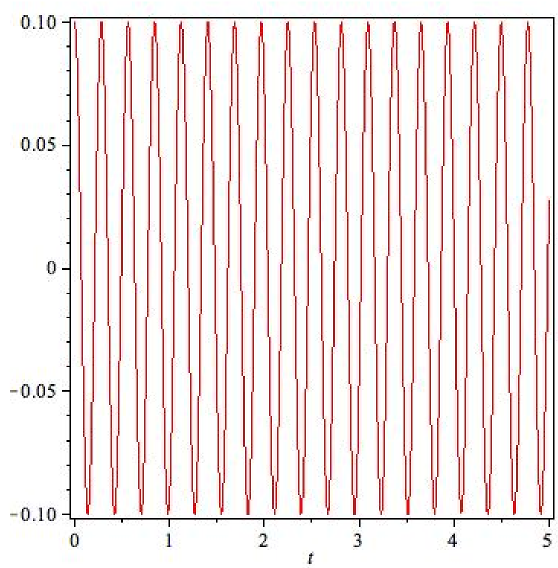
$$x(0) = 0.1, \quad v(0) = 0.$$

Problem 4 Using the value of k that you derived on the first page, find the solution to our initial value problem above by finding the parameters of a function of the form

$$x(t) = A\sin(\alpha t + \beta) + C$$

which give a solution.

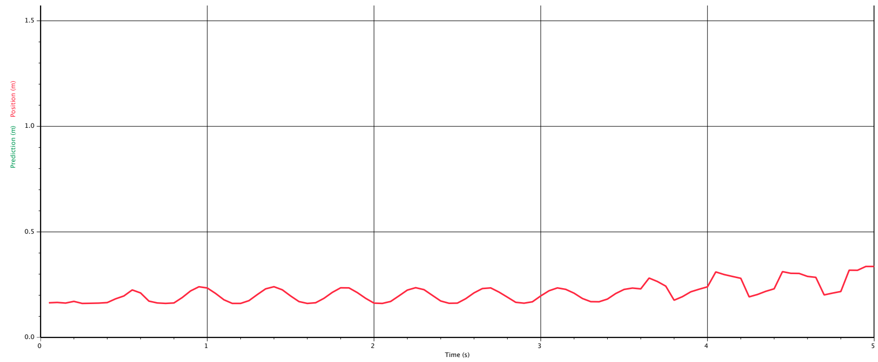
Here is a graph of the real solution:



How is this solution different from curve that you generated by Euler's method with $h = .001$?

Back To Reality

The following data was collected using a Go-Motion Sensor. The sensor was placed 25 centimeters below the equilibrium point (the spring was hanging above the sensor), therefore the curve is upsidedown and shifted.Also as you can see from the graph the spring was not released immediately. We see that the harmonic motion gets disrupted after about 3.5 seconds because of the friction between the top of the spring and the rod on which it was hanging.



Note that the differential equations describing the motion of the spring, depend only on the distance the spring is stretched beyond the equilibrium point. Therefore if we stretch a slinky with no mass attached and let it go, we should see harmonic motion. The graph below shows the motion of a stretched slinky with no mass hanging from it. Obviously the harmonic motion gets dampened in this case and the motion eventually comes to a halt.

What are some of the forces that might cause the motion of the slinky to stop?