1. The function $f(x) = e^{2x} + x^3 + x$ is one-to-one (there is no need to check this). What is $(f^{-1})'(2 + e^2)$?

**Solution.** Because $f(x)$ is one-to-one, we know the inverse function exists. Recall that $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$. So our first step is to find $f^{-1}(2+e^2)$. By definition of the inverse functions $f^{-1}(2+e^2) = x$ if and only if $f(x) = 2+e^2$. So we solve for $x$:

$$f(x) = 2 + e^2$$
$$e^{2x} + x^3 + x = 2 + e^2.$$ Comparing coefficients, we get $x = 1$.

The next step is to calculate $f'(x)$:

$$f(x) = e^{2x} + x^3 + x$$
$$\Rightarrow f'(x) = 2e^{2x} + 3x^2 + 1.$$ So $f'(f^{-1}(2 + e^2)) = f'(1) = 2e^2 + 3 + 1 = 2e^2 + 4$. Putting this into the equation $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$, we get:

$$(f^{-1})'(2 + e^2) = \frac{1}{2e^2 + 4}.$$ 

2. Use logarithmic differentiation to compute the derivative of the function

$$y = \frac{2^x(x^3 + 2)}{\sqrt{x - 1}}.$$ 

**Solution.** Taking the natural log of both sides of the equation, we get:

$$\ln y = \ln \left( \frac{2^x(x^3 + 2)}{\sqrt{x - 1}} \right)$$
$$= \ln(2^x) + \ln(x^3 + 2) - \ln((x - 1)^{1/2})$$
$$= x \ln 2 + \ln(x^3 + 2) - \frac{1}{2} \ln(x - 1).$$
Differentiating both sides we obtain:

\[
\frac{1}{y} \cdot \frac{dy}{dx} = \ln 2 + \frac{3x^2}{x^3 + 2} - \frac{1}{2(x - 1)}
\]

\[\Rightarrow \frac{dy}{dx} = y \left( \ln 2 + \frac{3x^2}{x^3 + 2} - \frac{1}{2(x - 1)} \right).\]

Thus,

\[
\frac{dy}{dx} = \frac{2^x(x^3 + 2)}{\sqrt{x - 1}} \left( \ln 2 + \frac{3x^2}{x^3 + 2} - \frac{1}{2(x - 1)} \right).
\]

3. Compute the integral

\[
\int_1^5 \frac{1}{(1 + x^2) \tan^{-1}(x)} \, dx.
\]

**Solution.** We use u-substitution with \( u = \tan^{-1}(x) \Rightarrow du = \frac{dx}{1 + x^2} \). We must also change the bounds of integration:

\[x = 5 \rightarrow u = \tan^{-1}(5),\]

\[x = 1 \rightarrow u = \tan^{-1}(1) = \pi/4.\]

Doing these substitutions, the integral changes to:

\[
\int_{\pi/4}^{\tan^{-1}(5)} \frac{1}{u} \, du = \ln |u| \bigg|_{\pi/4}^{\tan^{-1}(5)} = \ln |\tan^{-1}(5)| - \ln(\pi/4).
\]

Note that \( \tan^{-1}(5) \) will be positive because 5 > 0, so the absolute value sign isn’t needed. So,

\[
\int_1^5 \frac{1}{(1 + x^2) \tan^{-1}(x)} \, dx = \ln(\tan^{-1}(5)) - \ln(\pi/4).
\]
4. Find the derivative of \((x^2 + 1)^{x^2+1}\).

**Solution.** Since there is a variable in both the base and the exponent, we need logarithmic differentiation. Let \(y = (x^2 + 1)^{x^2+1}\). Then
\[
\ln y = \ln \left((x^2 + 1)^{x^2+1}\right) \\
= (x^2 + 1) \cdot \ln (x^2 + 1).
\]
Differentiating,
\[
\frac{1}{y} y' = 2x \ln (x^2 + 1) + (x^2 + 1) \frac{2x}{x^2 + 1} \\
= 2x \ln (x^2 + 1) + 2x \\
= 2x (\ln (x^2 + 1) + 1).
\]
Multiplying both sides by \(y = (x^2 + 1)^{x^2+1}\), we obtain
\[
y' = (x^2 + 1)^{x^2+1} 2x (\ln (x^2 + 1) + 1).
\]

5. Which of the following is true about \(y = f(x) = x \ln(x)\), \(x > 0\)?

(a) The function is decreasing for \(0 < x < \frac{1}{e}\), increasing for \(x > \frac{1}{e}\), and concave up for all \(x > 0\).
(b) The function is increasing for all \(x\) and concave up for all \(x > 0\).
(c) The function is decreasing for \(0 < x < e\), increasing for \(x > e\), and concave up for all \(x > 0\).
(d) The function is decreasing for \(0 < x < 1\), increasing for \(x > 1\), and concave up for all \(x > 0\).
(e) The function is concave down for all \(x > 0\).

**Solution.** Consider the first derivative
\[
y' = \ln x + 1.
\]
We note that
\[
\ln x + 1 < 0 \iff \ln x < -1 \iff x < e^{-1} = \frac{1}{e}.
\]
So the function \(y\) is decreasing on the interval \(0 < x < \frac{1}{e}\). Similarly,
\[
\ln x + 1 > 0 \iff \ln x > -1 \iff x > e^{-1} = \frac{1}{e},
\]
so the function \(y\) is increasing on the interval \(x > \frac{1}{e}\). To determine concavity, we look at the second derivative
\[
y'' = \frac{1}{x}.
\]
Since $\frac{1}{x}$ is positive for $x > 0$, the function $y$ is concave up for all $x > 0$.

**Answer:** The function is decreasing for $0 < x < \frac{1}{e}$, increasing for $x > \frac{1}{e}$, and concave up for all $x > 0$.

6. Compute the following definite integral.

$$\int_0^{\log_5(6)} \frac{5^x}{\sqrt{1 + 5^x}} \, dx.$$ 

**Solution.** We choose the substitution $u = 5^x + 1$. Then $du = 5^x \ln 5 \, dx$ and

$$\int_0^{\log_5(6)} \frac{5^x}{\sqrt{1 + 5^x}} \, dx = \frac{1}{\ln 5} \int_2^7 \frac{1}{\sqrt{u}} \, du = \frac{2\sqrt{u}}{\ln 5} \bigg|_2^7 = \frac{2(\sqrt{7} - \sqrt{2})}{\ln 5}.$$
7. Find the limit

$$\lim_{x \to 0} \frac{\ln(1 + 2x)}{\sin(x)}.$$ 

**Solution.** Plugging $x = 0$ into the expression $\frac{\ln(1 + 2x)}{\sin(x)}$ we see that the limit is of indeterminate form $\frac{0}{0}$. We apply L’Hospital’s Rule:

$$\lim_{x \to 0} \frac{\ln(1 + 2x)}{\sin(x)} = \lim_{x \to 0} \frac{\frac{d}{dx}[\ln(1 + 2x)]}{\frac{d}{dx}[\sin(x)]}$$

$$= \lim_{x \to 0} \frac{\frac{2}{1 + 2x}}{\cos(x)}$$

$$= \lim_{x \to 0} \frac{2}{(\cos(x))(1 + 2x)}$$

$$= \frac{2}{\cos(0) \cdot 1}$$

$$= 2.$$ 

8. Find the integral

$$\int_0^1 e^x x^2 \, dx.$$ 

**Solution.** We need to use integration by parts, with $u = x^2$, $dv = e^x \, dx$. Then $du = 2xdx$ and $v = e^x$. (We make this choice because we want to decrease the power of $x$.) Now,

$$\int_0^1 e^x x^2 \, dx = x^2 e^x \bigg|_0^1 - 2 \int_0^1 xe^x \, dx$$

$$= (e^1 - 0) - 2 \int_0^1 xe^x \, dx$$

$$= e - 2 \int_0^1 xe^x \, dx.$$
To evaluate the integral $\int_0^1 xe^x \, dx$, we again need to use integration by parts. This time we choose $u = x$, $dv = e^x \, dx$. So $du = dx$ and $v = e^x$. Thus,

$$\int_0^1 xe^x \, dx = xe^x \bigg|_0^1 - \int_0^1 e^x \, dx$$

$$= (e - 0) - e^1 \bigg|_0^1$$

$$= e - (e - 1)$$

$$= 1.$$

Substituting this back into the above we obtain

$$\int_0^1 e^x x^2 \, dx = e - 2 \int_0^1 xe^x \, dx$$

$$= e - 2 \cdot 1$$

$$= e - 2.$$

**Remark:** If you choose to drop the bounds and first evaluate the indefinite integral $\int e^x x^2 \, dx$, this is fine as long as your solution is written in a mathematically correct way (i.e. do not have a definite integral equal to an indefinite integral). An example of how to do this would be as follows. Applying integration by parts twice (with the same choices for $u$ and $dv$ as above) we obtain:

$$\int e^x x^2 \, dx = x^2 e^x - 2 \int xe^x \, dx$$

$$= x^2 e^x - 2 \left[ xe^x - \int e^x \, dx \right]$$

$$= x^2 e^x - 2 (xe^x - e^x) + C$$

$$= x^2 e^x - 2xe^x + 2e^x + C.$$

Therefore,

$$\int_0^1 e^x x^2 \, dx = x^2 e^x - 2xe^x + 2e^x \bigg|_0^1$$

$$= (e^1 - 2e^1 + 2e^1) - (0 - 0 + 2e^0)$$

$$= e - 2.$$
9. Compute the integral
\[ \int_0^{\pi/4} \tan^3(x) \sec^3(x) \, dx. \]

**Solution.** This is an integral of the form \( \int \sec^m x \tan^n x \, dx \). Since both \( m \) and \( n \) are odd, we use the \( u \)-substitution \( u = \sec x \), \( du = \sec x \tan x \) and use the trigonometric identity \( 1 + \tan^2 x = \sec^2 x \) to rewrite \( \tan^2 x \) as \( \sec^2 x - 1 \).

Putting this all together, we obtain:

\[
\int_0^{\pi/4} \tan^3 x \ sec^3 x \, dx = \int_0^{\pi/4} (\tan^2 x \ sec^2 x)(\sec x \tan x) \, dx \\
= \int_0^{\pi/4} (\sec^2 x - 1)(\sec^2 x)(\sec x \tan x) \, dx \\
= \int_{\sec(0)}^{\sec(\pi/4)} (u^2 - 1) \cdot u^2 \, du \\
= \int_1^{\sqrt{2}} u^4 - u^2 \, du \\
= \left[ \frac{u^5}{5} - \frac{u^3}{3} \right]_1^\sqrt{2} \\
= \left( \frac{(\sqrt{2})^5}{5} - \frac{(\sqrt{2})^3}{3} \right) - \left( \frac{1}{5} - \frac{1}{3} \right) \\
= \frac{2^{5/2} - 1}{5} - \frac{2^{3/2} - 1}{3}.
\]

10. Compute the integral
\[ \int_0^2 \frac{x}{x^2 + \sqrt[3]{x}} \, dx. \]

**Hint:** a rationalizing substitution might help.

**Solution.** We can rewrite our integral as

\[
\int_0^2 \frac{x}{x^2 + \sqrt[3]{x}} \, dx = \int_0^2 \frac{x}{x^{2/3}(x^{5/3} + 1)} \, dx \\
= \int_0^2 \frac{x^{2/3}}{x^{5/3} + 1} \, dx.
\]
We now see that we can use the $u$-substitution $u = x^{5/3} + 1$, $du = \frac{5}{3}x^{2/3}\,dx$.

Next, we change the bounds of integration:

\[
x = 0 \Rightarrow u = 0^{5/3} + 1 = 1,
\]
\[
x = 2 \Rightarrow u = 2^{5/3} + 1 = \frac{3\sqrt{32}}{32} + 1.
\]

Putting this together, we obtain

\[
\int_0^2 \frac{x}{x^2 + \sqrt{x}} \,dx = \int_0^2 \frac{x^{2/3}}{x^{5/3} + 1} \,dx
\]
\[
= \frac{3}{5} \int_1^{\sqrt[5]{32} + 1} \frac{du}{u}
\]
\[
= \frac{3}{5} \ln |u|^{\sqrt[5]{32} + 1}
\]
\[
= \frac{3 \ln(\sqrt[5]{32} + 1)}{5}.
\]

11. The population (in thousands (= th.)) of UNClan had an initial value of 100 th. at time $t = 0$ days. The population of UNClan has been decreasing since then at a rate proportional to its size. The population of UNClan was 60 th. at time $t = 20$ days. Let $P(t)$ denote the size of the population of UNClan $t$ days after $t = 0$.

(a) Find an expression for $P(t)$ in the form

\[P(t) = Ce^{kt},\]

where $C$ and $k$ are constants. (You should NOT attempt to approximate numbers such as $\ln(2)$ etc. in decimal or fractional form.)

Solution. At $t = 0$, we have: $P(0) = 100 = Ce^{k\cdot0} = Ce^0 = C$, so $C = 100$.

At $t = 20$, we have:

\[P(20) = Ce^{20k},
\]
\[60 = 100e^{20k},
\]
\[\frac{60}{100} = \frac{3}{5} = e^{20k},
\]
Taking the natural logarithm of both sides:

\[
\ln \left( \frac{3}{5} \right) = \ln(e^{20k}) \\
\ln(3) - \ln(5) = 20k \\
\frac{\ln(3) - \ln(5)}{20} = k.
\]

Answer: \[ C = 100 \text{th., } k = \frac{\ln 3 - \ln 5}{20} \]

(b) When will the population of UNClan equal 10 th.? (You may write your answer in terms of logarithms.)

Solution. We want to know for which \( t \) we get \( P(t) = 10 \):

\[ P(t) = 100e^{kt} = 10 \]

\[ \Rightarrow e^{\frac{\ln 3 - \ln 5}{20} \cdot t} = \frac{10}{100} = \frac{1}{10} \]

Take the natural logarithm of both sides:

\[ \frac{\ln 3 - \ln 5}{20} \cdot t = \ln(1/10) = -\ln 10 \]

\[ t = \frac{-20 \ln(10)}{\ln(3) - \ln(5)} = \frac{20 \ln(10)}{\ln(5) - \ln(3)} \]

So the population of UNClan is 10 th. when

\[ t = \frac{20 \ln(10)}{\ln(5) - \ln(3)} \text{ days.} \]

Equivalent Answers: \[\frac{20 \ln(1/10)}{\ln(3/5)} \text{ days, } \frac{20 \ln(10)}{\ln(5/3)} \text{ days.}\]

12. Compute the integral

\[ \int \frac{x^2 + 1}{x^3 + x^2} \, dx. \]

Solution. We proceed using partial fraction decomposition. First, we factor the denominator

\[ x^3 + x^2 = x^2(x + 1). \]
Since the linear factor $x$ occurs twice, the partial fraction decomposition is

$$\frac{x^2 + 1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}.$$ 

Multiplying both sides by $x^2(x + 1)$, we get

$$x^2 + 1 = Ax(x + 1) + B(x + 1) + Cx^2$$

$$= (A + C)x^2 + (A + B)x + B.$$ 

Equating coefficients

$1 = B$

$0 = A + B$

$1 = A + C$

we obtain $A = -1$, $B = 1$, and $C = 2$. Now, returning to the integral,

$$\int \frac{x^2 + 1}{x^2(x + 1)} \, dx = \int -\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x + 1} \, dx$$

$$= -\ln |x| - \frac{1}{x} + 2 \ln |x + 1| + C.$$ 

13. Compute

$$\int \frac{1}{\sqrt{x^2 - 2x + 2}} \, dx$$

Present your answer as a function of the variable $x$. Note the formula sheet at the back of the exam may be helpful in working out your final answer.

**Solution.** Completing the square we see that $x^2 - 2x + 2 = x^2 - 2x + 1^2 - 1^2 + 2 = (x - 1)^2 + 1$. Thus we can rewrite our integral as

$$\int \frac{1}{\sqrt{x^2 - 2x + 2}} \, dx = \int \frac{1}{\sqrt{(x - 1)^2 + 1}} \, dx$$

$$= \int \frac{1}{\sqrt{u^2 + 1}} \, du \quad \text{(Substitution: } u = x - 1, \, du = dx)$$

Next, we want to make a trigonometric substitution. Because our expression is of the form $u^2 + a^2$, we use the substitution $u = \tan \theta, \, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. So $du = \sec^2 \theta \, d\theta$, and our integral becomes
\[\int \frac{1}{\sqrt{x^2 - 2x + 2}} \, dx = \int \frac{1}{\sqrt{u^2 + 1}} \, du \]
\[= \int \frac{1}{\sqrt{\tan^2 \theta + 1}} \cdot \sec^2 \theta \, d\theta \]
\[= \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta}} \, d\theta \]
\[= \int \sec \theta \, d\theta \]
\[= \ln |\sec \theta + \tan \theta| + C.\]

Next we need to substitute back in for \( \theta \). We had \( u = \tan \theta \), where \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\). Further, since \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\), we have \( \sec \theta = \frac{1}{\cos \theta} > 0 \). Now, \( \sec^2 \theta = 1 + \tan^2 \theta = 1 + u^2 \), so taking the positive square root, \( \sec \theta = \sqrt{u^2 + 1} \).

Now,
\[\int \frac{1}{\sqrt{x^2 - 2x + 2}} \, dx = \ln |\sec \theta + \tan \theta| + C \]
\[= \ln |\sqrt{u^2 + 1} + u| + C \]

Finally, we had \( u = x - 1 \) and \( \sqrt{u^2 + 1} = \sqrt{x^2 - 2x + 2} \), so substituting this back in we obtain
\[\int \frac{1}{\sqrt{x^2 - 2x + 2}} \, dx = \ln |\sqrt{x^2 - 2x + 2} + x - 1| + C.\]