Solutions to Exam 1, Math 10560

1. The function $f(x) = e^{2x} + x^3 + x$ is one-to-one (there is no need to check this). What is $(f^{-1})'(2 + e^2)$?

Solution. Because $f(x)$ is one-to-one, we know the inverse function exists. Recall that $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$. So our first step is to find $f^{-1}(2+e^2)$. By definition of the inverse functions $f^{-1}(2+e^2) = x$ if and only if $f(x) = 2+e^2$. So we solve for x:

$$\begin{aligned} f(x) &= 2 + e^2 \\ e^{2x} + x^3 + x &= 2 + e^2. \end{aligned}$$

Comparing coefficients, we get $x = 1$.

The next step is to calculate $f'(x)$:

$$\begin{aligned} f(x) &= e^{2x} + x^3 + x \\ \Rightarrow f'(x) &= 2e^{2x} + 3x^2 + 1. \end{aligned}$$

So $f'(f^{-1}(2 + e^2)) = f'(1) = 2e^2 + 3 + 1 = 2e^2 + 4$. Putting this into the equation $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$, we get:

$$\boxed{(f^{-1})'(2 + e^2) = \frac{1}{2e^2 + 4}.}$$

2. Use logarithmic differentiation to compute the derivative of the function

$$y = \frac{2^x(x^3 + 2)}{\sqrt{x - 1}}.$$

Solution. Taking the natural log of both sides of the equation, we get:

$$\begin{aligned} \ln y &= \ln\left(\frac{2^x(x^3 + 2)}{\sqrt{x - 1}}\right) \\ &= \ln(2^x) + \ln(x^3 + 2) - \ln((x - 1)^{1/2}) \\ &= x \ln 2 + \ln(x^3 + 2) - \frac{1}{2}\ln(x - 1). \end{aligned}$$

Differentiating both sides we obtain:

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln 2 + \frac{3x^2}{x^3+2} - \frac{1}{2(x-1)}$$
$$\Rightarrow \frac{dy}{dx} = y\left(\ln 2 + \frac{3x^2}{x^3+2} - \frac{1}{2(x-1)}\right).$$

Thus,

$$\boxed{\frac{dy}{dx} = \frac{2^x(x^3+2)}{\sqrt{x-1}}\left(\ln 2 + \frac{3x^2}{x^3+2} - \frac{1}{2(x-1)}\right).}$$

3. Compute the integral

$$\int_1^5 \frac{1}{(1+x^2)\tan^{-1}(x)}dx.$$

Solution. We use u-substitution with $u = \tan^{-1}(x) \Rightarrow du = \frac{dx}{1+x^2}$. We must also change the bounds of integration:

$$x = 5 \to u = \tan^{-1}(5),$$
$$x = 1 \to u = \tan^{-1}(1) = \pi/4.$$

Doing these substitutions, the integral changes to:

$$\int_{\pi/4}^{\tan^{-1}(5)} \frac{1}{u}du = \ln|u|\Bigg|_{\pi/4}^{\tan^{-1}(5)} = \ln|\tan^{-1}(5)| - \ln(\pi/4).$$

Note that $\tan^{-1}(5)$ will be positive because $5 > 0$, so the absolute value sign isn't needed. So,

$$\int_1^5 \frac{1}{(1+x^2)\tan^{-1}(x)}dx = \boxed{\ln(\tan^{-1}(5)) - \ln(\pi/4).}$$

4. Find the derivative of $(x^2+1)^{x^2+1}$.

Solution. Since there is a variable in both the base and the exponent, we need logarithmic differentiation. Let $y=(x^2+1)^{x^2+1}$. Then

$$\begin{aligned}\ln y &= \ln\left((x^2+1)^{x^2+1}\right)\\ &= (x^2+1)\cdot\ln(x^2+1).\end{aligned}$$

Differentiating,

$$\begin{aligned}\frac{1}{y}\,y' &= 2x\ln(x^2+1)+(x^2+1)\frac{2x}{x^2+1}\\ &= 2x\ln(x^2+1)+2x\\ &= 2x\left(\ln(x^2+1)+1\right).\end{aligned}$$

Multiplying both sides by $y=(x^2+1)^{x^2+1}$, we obtain

$$\boxed{y'=(x^2+1)^{x^2+1}\,2x\left(\ln(x^2+1)+1\right).}$$

5. Which of the following is true about $y=f(x)=x\ln(x)$, $x>0$?

(a) The function is decreasing for $0<x<\frac{1}{e}$, increasing for $x>\frac{1}{e}$, and concave up for all $x>0$.
(b) The function is increasing for all x and concave up for all $x>0$.
(c) The function is decreasing for $0<x<e$, increasing for $x>e$, and concave up for all $x>0$.
(d) The function is decreasing for $0<x<1$, increasing for $x>1$, and concave up for all $x>0$.
(e) The function is concave down for all $x>0$.

Solution. Consider the first derivative

$$y'=\ln x+1.$$

We note that

$$\ln x+1<0 \iff \ln x<-1 \iff x<e^{-1}=\frac{1}{e}.$$

So the function y is decreasing on the interval $0<x<\frac{1}{e}$. Similarly,

$$\ln x+1>0 \iff \ln x>-1 \iff x>e^{-1}=\frac{1}{e},$$

so the function y is increasing on the interval $x>\frac{1}{e}$. To determine concavity, we look at the second derivative

$$y''=\frac{1}{x}.$$

Since $\frac{1}{x}$ is positive for $x > 0$, the function y is concave up for all $x > 0$.

Answer: $\boxed{\text{The function is decreasing for } 0 < x < \frac{1}{e}, \text{ increasing for } x > \frac{1}{e}, \text{ and concave up for all } x > 0.}$

6. Compute the following definite integral.

$$\int_0^{\log_5(6)} \frac{5^x}{\sqrt{1+5^x}}\, dx.$$

Solution. We choose the substitution $u = 5^x + 1$. Then $du = 5^x \ln 5\, dx$ and

$$\begin{aligned}\int_0^{\log_5(6)} \frac{5^x}{\sqrt{1+5^x}}\, dx &= \frac{1}{\ln 5}\int_2^7 \frac{1}{\sqrt{u}}\, du \\ &= \left.\frac{2\sqrt{u}}{\ln 5}\right|_2^7 \\ &= \boxed{\frac{2\left(\sqrt{7}-\sqrt{2}\right)}{\ln 5}}.\end{aligned}$$

7. Find the limit

$$\lim_{x\to 0} \frac{\ln(1+2x)}{\sin(x)}.$$

Solution. Plugging $x = 0$ into the expression $\frac{\ln(1+2x)}{\sin(x)}$ we see that the limit is of indeterminate form $\frac{0}{0}$. We apply L'Hospital's Rule:

$$\begin{aligned}
\lim_{x\to 0} \frac{\ln(1+2x)}{\sin(x)} &= \lim_{x\to 0} \frac{\frac{d}{dx}[\ln(1+2x)]}{\frac{d}{dx}[\sin(x)]} \\
&= \lim_{x\to 0} \frac{\frac{2}{1+2x}}{\cos(x)} \\
&= \lim_{x\to 0} \frac{2}{(\cos(x))(1+2x)} \\
&= \frac{2}{\cos(0)\cdot 1} \\
&= \boxed{2.}
\end{aligned}$$

8. Find the integral

$$\int_0^1 e^x x^2\,dx.$$

Solution. We need to use integration by parts, with $u = x^2$, $dv = e^x\,dx$. Then $du = 2xdx$ and $v = e^x$. (We make this choice because we want to decrease the power of x.) Now,

$$\begin{aligned}
\int_0^1 e^x x^2\,dx &= x^2e^x\Big|_0^1 - 2\int_0^1 xe^x\,dx \\
&= (e^1 - 0) - 2\int_0^1 xe^x\,dx \\
&= e - 2\int_0^1 xe^x\,dx.
\end{aligned}$$

To evaluate the integral $\int_0^1 xe^x\,dx$, we again need to use integration by parts. This time we choose $u = x$, $dv = e^x\,dx$. So $du = dx$ and $v = e^x$. Thus,

$$\begin{aligned}\int_0^1 xe^x\,dx &= xe^x\Big|_0^1 - \int_0^1 e^x\,dx \\ &= (e-0) - e^x\Big|_0^1 \\ &= \cancel{e} - (\cancel{e} - 1) \\ &= 1.\end{aligned}$$

Substituting this back into the above we obtain

$$\begin{aligned}\int_0^1 e^x x^2\,dx &= e - 2\int_0^1 xe^x\,dx \\ &= e - 2\cdot 1 \\ &= \boxed{e-2\,.}\end{aligned}$$

Remark: If you choose to drop the bounds and first evaluate the indefinite integral $\int e^x x^2\,dx$, this is fine as long as your solution is written in a mathematically correct way (i.e. do not have a definite integral equal to an indefinite integral). An example of how to do this would be as folllows. Applying integration by parts twice (with the same choices for u and dv as above) we obtain:

$$\begin{aligned}\int e^x x^2\,dx &= x^2e^x - 2\int xe^x\,dx \\ &= x^2e^x - 2\left[xe^x - \int e^x\,dx\right] \\ &= x^2e^x - 2\left(xe^x - e^x\right) + C \\ &= x^2e^x - 2xe^x + 2e^x + C.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 e^x x^2 &= x^2e^x - 2xe^x + 2e^x\Big|_0^1 \\ &= (e^1 - \cancel{2e^1} + \cancel{2e^1}) - (0 - 0 + 2e^0) \\ &= \boxed{e-2.}\end{aligned}$$

9. Compute the integral

$$\int_0^{\pi/4} \tan^3(x)\ \sec^3(x)\,dx.$$

Solution. This is an integral of the form $\int \sec^m x \tan^n x\,dx$. Since both m and n are odd, we use the u-substitution $u = \sec x$, $du = \sec x \tan x$ and use the trigonometric identity $1 + \tan^2 x = \sec^2 x$ to rewrite $\tan^2 x$ as $\sec^2 x - 1$. Putting this all together, we obtain:

$$\begin{aligned}
\int_0^{\pi/4} \tan^3 x\ \sec^3 x\,dx &= \int_0^{\pi/4} (\tan^2 x \sec^2 x)(\sec x \tan x)\,dx \\
&= \int_0^{\pi/4} (\sec^2 x - 1)(\sec^2 x)(\sec x \tan x)\,dx \\
&= \int_{\sec(0)}^{\sec(\pi/4)} (u^2 - 1) \cdot u^2 du \\
&= \int_1^{\sqrt{2}} u^4 - u^2\,du \\
&= \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^{\sqrt{2}} \\
&= \left(\frac{(\sqrt{2})^5}{5} - \frac{(\sqrt{2})^3}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right) \\
&= \boxed{\frac{2^{5/2} - 1}{5} - \frac{2^{3/2} - 1}{3}.}
\end{aligned}$$

10. Compute the integral

$$\int_0^2 \frac{x}{x^2 + \sqrt[3]{x}}\,dx.$$

Hint: a rationalizing substitution might help.

Solution. We can rewrite our integral as

$$\begin{aligned}
\int_0^2 \frac{x}{x^2 + \sqrt[3]{x}}\,dx &= \int_0^2 \frac{x}{x^{1/3}(x^{5/3} + 1)}\,dx \\
&= \int_0^2 \frac{x^{2/3}}{x^{5/3} + 1}\,dx.
\end{aligned}$$

We now see that we can use the u-substitution $u = x^{5/3} + 1$, $du = \frac{5}{3}x^{2/3}\,dx$. Next, we change the bounds of integration:

$$x = 0 \Rightarrow u = 0^{5/3} + 1 = 1,$$
$$x = 2 \Rightarrow u = 2^{5/3} + 1 = \sqrt[3]{32} + 1.$$

Putting this together, we obtain

$$\begin{aligned}
\int_0^2 \frac{x}{x^2 + \sqrt[3]{x}} &= \int_0^2 \frac{x^{2/3}}{x^{5/3} + 1}\,dx \\
&= \frac{3}{5}\int_1^{\sqrt[3]{32}+1} \frac{du}{u} \\
&= \frac{3}{5}\ln|u|\Big|_1^{\sqrt[3]{32}+1} \\
&= \boxed{\frac{3\ln(\sqrt[3]{32}+1)}{5}.}
\end{aligned}$$

11. The population (in thousands (= th.)) of UNCland had an initial value of 100 th. at time $t = 0$ days. The population of UNCland has been decreasing since then at a rate proportional to its size. The population of UNCland was 60 th. at time $t = 20$ days. Let $P(t)$ denote the size of the population of UNCland t days after $t = 0$.

(a) Find an expression for $P(t)$ in the form

$$P(t) = Ce^{kt},$$

where C and k are constants. (You should **NOT** attempt to approximate numbers such as ln(2) etc.. in decimal or fractional form.)

Solution. At $t = 0$, we have: $P(0) = 100 = Ce^{k\cdot 0} = Ce^0 = C$, so $C = 100$. At $t = 20$, we have:

$$\begin{aligned}
P(20) &= Ce^{20k} \\
60 &= 100e^{20k} \\
\frac{60}{100} = \frac{3}{5} &= e^{20k},
\end{aligned}$$

Taking the natural logarithm of both sides:

$$\begin{aligned}
\ln\left(\frac{3}{5}\right) &= \ln(e^{20k}) \\
\ln(3) - \ln(5) &= 20k \\
\frac{\ln(3) - \ln(5)}{20} &= k.
\end{aligned}$$

Answer: $\boxed{C = 100\text{th.}, k = \frac{\ln 3 - \ln 5}{20}}$

(b) When will the population of UNCland equal 10 th. ?
(You may write your answer in terms of logarithms.)

Solution. We want to know for which t we get $P(t) = 10$:

$$\begin{aligned}
P(t) &= 100e^{kt} = 10 \\
&\Rightarrow e^{\frac{\ln 3 - \ln 5}{20} \cdot t} = \frac{10}{100} = \frac{1}{10}
\end{aligned}$$

Take the natural logarithm of both sides:

$$\begin{aligned}
\frac{\ln 3 - \ln 5}{20} \cdot t &= \ln(1/10) = -\ln 10 \\
t &= \frac{-20\ln(10)}{\ln(3) - \ln(5)} = \frac{20\ln(10)}{\ln(5) - \ln(3)}
\end{aligned}$$

So the population of UNCland is 10 th. when

$$\boxed{t = \frac{20\ln(10)}{\ln(5) - \ln(3)} \text{ days.}}$$

Equivalent Answers: $\frac{20\ln(1/10)}{\ln(3/5)}$ days, $\frac{20\ln(10)}{\ln(5/3)}$ days.

12. Compute the integral

$$\int \frac{x^2 + 1}{x^3 + x^2}\, dx.$$

Solution. We proceed using partial fraction decomposition. First, we factor the denominator

$$x^3 + x^2 = x^2(x + 1).$$

Since the linear factor x occurs twice, the partial fraction decomposition is

$$\frac{x^2+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$

Multiplying both sides by $x^2(x+1)$, we get

$$\begin{aligned} x^2+1 &= Ax(x+1) + B(x+1) + Cx^2 \\ &= (A+C)x^2 + (A+B)x + B. \end{aligned}$$

Equating coefficients

$$\begin{aligned} 1 &= B \\ 0 &= A+B \\ 1 &= A+C \end{aligned}$$

we obtain $A = -1$, $B = 1$, and $C = 2$. Now, returning to the integral,

$$\begin{aligned} \int \frac{x^2+1}{x^2(x+1)}\,dx &= \int -\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x+1}\,dx \\ &= \boxed{-\ln|x| - \frac{1}{x} + 2\ln|x+1| + C.} \end{aligned}$$

13. Compute

$$\int \frac{1}{\sqrt{x^2-2x+2}}\,dx$$

Present your answer as a function of the variable x. Note the formula sheet at the back of the exam may be helpful in working out your final answer.

Solution. Completing the square we see that $x^2-2x+2 = x^2-2x+1^2-1^2+2 = (x-1)^2+1$. Thus we can rewrite our integral as

$$\begin{aligned} \int \frac{1}{\sqrt{x^2-2x+2}}\,dx &= \int \frac{1}{\sqrt{(x-1)^2+1}}\,dx \\ &= \int \frac{1}{\sqrt{u^2+1}}\,du \qquad \text{(Substitution: } u = x-1,\ du = dx) \end{aligned}$$

Next, we want to make a trigonometric substitution. Because our expression is of the form u^2+a^2, we use the substitution $u = \tan\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. So $du = \sec^2\theta\,d\theta$, and our integral becomes

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-2x+2}}\,dx &= \int \frac{1}{\sqrt{u^2+1}}\,du \\
&= \int \frac{1}{\sqrt{\tan^2\theta+1}} \cdot \sec^2\theta\,d\theta \\
&= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta \\
&= \int \sec\theta d\theta \\
&= \ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

Next we need to substitute back in for θ. We had $u = \tan\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Further, since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we have $\sec\theta = \frac{1}{\cos\theta} > 0$. Now, $\sec^2\theta = 1+\tan^2\theta = 1+u^2$, so taking the positive square root, $\sec\theta = \sqrt{u^2+1}$. Now,

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-2x+2}} &= \ln|\sec\theta+\tan\theta| + C \\
&= \ln|\sqrt{u^2+1}+u| + C
\end{aligned}$$

Finally, we had $u = x-1$ and $\sqrt{u^2+1} = \sqrt{x^2-2x+2}$, so substituting this back in we obtain

$$\boxed{\int \frac{1}{\sqrt{x^2-2x+2}}\,dx = \ln|\sqrt{x^2-2x+2}+x-1| + C.}$$