Math 10560, Exam 3  
April 29, 2015

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 11 pages of the test.

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Multiple Choice _____________

11. _______________
12. _______________
13. _______________

Total _______________
Multiple Choice

1. (6 pts) Find

\[ \lim_{n \to \infty} n \tan \left( \frac{1}{2n} \right) \]

(a) \( \frac{1}{2} \) (b) 2 (c) \( \infty \) (d) 0 (e) \( \frac{1}{4} \)

**Sol.** Let \( x = 1/(2n) \) so that the limit becomes

\[ \lim_{n \to \infty} n \tan \left( \frac{1}{2n} \right) = \lim_{x \to 0} \frac{1}{2x} \tan \left( x \right) = \lim_{x \to 0} \frac{1}{2x} \frac{\sin(x)}{\cos(x)} = \frac{1}{2}. \]

2. (6 pts) Find the sum of the following series.

\[ \sum_{k=0}^{\infty} \frac{(-1)^{k+1} + 3^k}{4^k} . \]

(a) \( \frac{3}{8} \) (b) \( \frac{16}{5} \) (c) \( \frac{24}{5} \) (d) \( \frac{8}{3} \) (e) 0

**Sol.** Splitting the series into two pieces, we have

\[ \sum_{k=0}^{\infty} \frac{(-1)^{k+1} + 3^k}{4^k} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^k} + \sum_{k=0}^{\infty} \frac{3^k}{4^k} . \]

The second series is geometric, and equals \( 1/(1 - 3/4) = 4 \). The first series is also geometric, after we write it as

\[ \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^k} = -\sum_{k=0}^{\infty} (-1/4)^k , \]

and therefore sums to \(-4/5\). Adding the two parts, we obtain the series sums to \( \frac{16}{5} \).
3. (6 pts) Which of the following statement is TRUE?

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n(\sqrt{n} + 1)}{n} \] converges absolutely.

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^n(\sqrt{n} + 1)}{n} \] diverges.

(c) \[ \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n} \] diverges by divergence test.

(d) \[ \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n} \] converges conditionally.

(e) \[ \sum_{n=1}^{\infty} \frac{(-1)^n(\sqrt{n} + 1)}{n} \] converges conditionally.

**Sol.** We see that by the alternating series test, \[ \sum_{n=1}^{\infty} \frac{(-1)^n(\sqrt{n} + 1)}{n} \] converges. However, it is not absolutely convergent, by the comparison test. Comparing \[ \frac{(-1)^n(\sqrt{n} + 1)}{n} \] to \[ \frac{\sqrt{n}}{n} \], we see that this series diverges. Thus, this is a conditionally convergent series.

4. (6 pts) Find the sum of the following series,

\[ \sum_{n=1}^{\infty} \left[ \frac{n}{e^{n-1}} - \frac{n+1}{e^n} \right] . \]

(a) 0 \hspace{1cm} (b) 1 \hspace{1cm} (c) \frac{2}{e} \hspace{1cm} (d) \frac{1}{e} \hspace{1cm} (e) the series diverges

**Sol.** This is a telescoping series where the individual terms tend towards zero as \( n \to \infty \). Therefore, it sums to the first term, which is 1.
5. (6 pts) Which one of the following statement is TRUE?

(a) \( \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1)n} \) is absolutely convergent by root test.

(b) \( \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1)n} \) is divergent by ratio test.

(c) \( \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1)n} \) is absolutely convergent by ratio test.

(d) \( \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1)n} \) is divergent by comparison test.

(e) none of the above

Solution (d)
(a) It does not make sense to use the roots test here.
(b) The \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \), so the ratio test is inconclusive.
(c) See (b)
(d) \( \sum_{n=1}^{\infty} \frac{1}{((\sin n)^2 + 1)n} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \), so the series diverges by the comparison test.

6. (6 pts) For what values of \( p \) is the following series convergent?

\[ \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \ln n}{n^p}. \]

(a) for all \( p \) \hspace{1cm} (b) \( p > 1 \) \hspace{1cm} (c) \( p > 0 \)

(d) \( p < 0 \) \hspace{1cm} (e) for any \( p \) such that \( p \neq 0 \)

Solution
Since this is an alternating series, we only need to apply the alternating series test. If \( p > 0 \) then \( |b_{n+1}| < |b_n| \), and \( \lim_{n \to \infty} \frac{\ln n}{n^p} = 0 \) if \( p > 0 \) and = \( \infty \) if \( p < 0 \), so the answer is (c).
7. (6 pts) Expand \( \frac{1}{2x-x^2} \) as a power series centered around \( a = 1 \).

**Hint:** Complete squares in the denominator, and use a well known power series.

(a) \( \sum_{n=0}^{\infty} (-1)^n (x - 1)^{2n} \)
(b) \( \sum_{n=0}^{\infty} (x - 2)^n \)
(c) \( \sum_{n=0}^{\infty} \frac{(x - 1)^n}{2^n} \)
(d) \( \sum_{n=0}^{\infty} (x - 1)^{2n} \)
(e) \( \sum_{n=0}^{\infty} (-1)^n (x - 1)^n \)

**Solution (d)**

If you use the hint, this problem is very fast.

\[
\frac{1}{2x-x^2} = \frac{1}{1 - (x-1)^2} = \sum_{n=0}^{\infty} (x - 1)^{2n}
\]

8. (6 pts) Use the MacLaurin series to find \( \lim_{x \to 0} \frac{e^{-x^2} - 1 + x^2}{x^4} \).

(a) \(-\frac{1}{6}\)
(b) \(-1\)
(c) \(\frac{1}{2}\)
(d) 0
(e) \(\frac{1}{3}\)

**Sol.** We have \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). Then

\[
e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}
\]

Therefore

\[
\lim_{x \to 0} \frac{e^{-x^2} - 1 + x^2}{x^4} = \lim_{x \to 0} \frac{1 - x^2 + \frac{x^4}{2} + O(x^6) - 1 + x^2}{x^4} = \lim_{x \to 0} \frac{x^4}{x^4} = \frac{1}{2}
\]
9. (6 pts) Find the radius of convergence of the following power series

\[ \sum_{n=1}^{\infty} \frac{n^n}{n!} x^n \]

Hint: \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \).

(a) 0 (b) \( \infty \) (c) \( e \) (d) 1 (e) \( e^{-1} \)

**Sol.** Let \( a_n = \frac{n^n}{n!} \). We have radius of convergence

\[ R = \lim_{n \to \infty} \frac{|a_n|}{a_{n+1}} = \lim_{n \to \infty} \frac{n^n}{n!} \frac{n!}{(n+1)^{n+1}} \]

\[ = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n \]

\[ = \lim_{n \to \infty} \left( \frac{1}{1 + \frac{1}{n}} \right)^n = \frac{1}{e} \]
10. (6 pts) Consider the function

\[ F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n2^n} \]

What is \( F^{(3)}(0) \)? Here \( F^{(3)} \) represents the third derivative.

(a) \( \frac{1}{4} \)  (b) 0  (c) 2  (d) \( \frac{1}{3} \)  (e) 6

**Sol.** We have

\[ F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{n2^n} \]

By equating the coefficient of \( x^3 \) we obtain

\[ \frac{F^{(3)}(0)}{6} = \frac{1}{24} \]

Thus

\[ F^{(3)}(0) = \frac{1}{4} \]
11. (12 pts.) Observe that \[ \int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) + C. \]

(a) Using this, find the Taylor series for \( \ln \left( \frac{1+x}{1-x} \right) \).

(b) Find the value of \[ \sum_{n=0}^{\infty} \frac{4^{-n}}{2n+1}. \]

Solution

(a) Using the observation:

\[ \int \frac{dx}{1-x^2} = \int \sum_{n=0}^{\infty} x^{2n} \, dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} + C = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right). \]

Setting \( x = 0 \) gives \( C = 0 \), so that

\[ 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \]

is the Taylor series expansion for \( \ln \left( \frac{1+x}{1-x} \right) \) about \( a = 0 \).

(b) \[ \sum_{n=0}^{\infty} \frac{4^{-n}}{2n+1} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^{2n} \frac{1}{2n+1} = 2 \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^{2n+1} \frac{1}{2n+1}. \]

The last equality here shows that we are plugging in \( x = 1/2 \) to the series obtained in part (a). Since the series in part (a) was the Taylor series for \( \ln \left( \frac{1+x}{1-x} \right) \), it follows that

\[ \sum_{n=0}^{\infty} \frac{4^{-n}}{2n+1} = \ln \left( \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right) = \ln 3. \]
12. (14 pts.) Test the following series for convergence. Specify the exact test being used, and check that all the required hypothesis are satisfied.

(a) \[ \sum_{n=1}^{\infty} \frac{1}{n - \sqrt{n} + 1} . \]

**Sol.** (a) We take \( a_n = \frac{1}{n - \sqrt{n} + 1} \) and \( b_n = \frac{1}{n} \).

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{n - \sqrt{n} + 1} = 1
\]

This combined with the fact that \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent shows that the series in question is divergent.

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n - \sqrt{n} + 1} . \]

**Sol.** (b) We note first

\[
\lim_{n \to \infty} \frac{1}{n - \sqrt{n} + 1} = 0
\]

and clearly \( \frac{1}{n - \sqrt{n} + 1} > 0 \).

Now let \( f(x) = \frac{1}{x - \sqrt{x} + 1} \).

\[
f'(x) = -\frac{1 - \frac{1}{2\sqrt{x}}}{(x - \sqrt{x} + 1)^2}
\]

When \( x > \frac{1}{4} \), \( f'(x) < 0 \). So \( f(x) \) is decreasing when \( x > \frac{1}{4} \).

So by alternating series test the series in question is convergent.
13. (14 pts.) Find the radius and interval of convergence of the following power series.

\[ \sum_{n=0}^{\infty} \frac{2^n(x - 1)^n}{n} \]

Be sure to discuss the convergence at the two end points.

**Sol.** Using the ratio test, we have

\[
L = \lim_{n \to \infty} \left| \frac{2^{n+1}(x - 1)^{n+1}}{n + 1} \cdot \frac{n}{2^n(x - 1)^n} \right| = \lim_{n \to \infty} \left| \frac{2n(x - 1)}{n + 1} \right| = 2|x - 1|.
\]

So we get that if \(2|x - 1| < 1 \implies |x - 1| < 1/2\) the series converges, and if \(|x - 1| > 1/2\) the series diverges. Thus the radius of convergence is \(R = 1/2\).

Now lets find the interval of convergence. We have that

\[-1/2 < x - 1 < 1/2 \implies 1/2 < x < 3/2.\]

Now lets check the endpoints. When \(x = 1/2\) we have

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n}, \]

which converges by the alternating series test.

When \(x = 3/2\) we have

\[ \sum_{n=0}^{\infty} \frac{1}{n}, \]

which is a harmonic series and diverges.

Thus the interval is

\[ 1/2 \leq x < 3/2. \]
The following is the list of useful trigonometric formulas:

\[
\sin^2 x + \cos^2 x = 1
\]

\[
1 + \tan^2 x = \sec^2 x
\]

\[
\sin^2 x = \frac{1}{2} (1 - \cos 2x)
\]

\[
\cos^2 x = \frac{1}{2} (1 + \cos 2x)
\]

\[
\sin 2x = 2 \sin x \cos x
\]

\[
\sin x \cos y = \frac{1}{2} \left( \sin(x - y) + \sin(x + y) \right)
\]

\[
\sin x \sin y = \frac{1}{2} \left( \cos(x - y) - \cos(x + y) \right)
\]

\[
\cos x \cos y = \frac{1}{2} \left( \cos(x - y) + \cos(x + y) \right)
\]

\[
\int \sec \theta = \ln | \sec \theta + \tan \theta | + C
\]
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Total 

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