Exam 1 Solutions.

1. We use the equation \((f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}\). Note that \(f'(x) = 3x^2e^{x^3} + 3\). We also need to compute \(f^{-1}(2)\). Since we can’t easily solve \(y = f(x)\) for \(x\), we try to guess a value of \(x\) so that \(f(x) = 2\); after a little trial-and-error we see that \(f(0) = 2\). Therefore,

\[
(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{3}.
\]

2. We first take a logarithm of \(y = f(x)\):

\[
\ln y = \ln \left( \frac{x^2 + 1}{\sqrt{x^3 + 2}} \right) = \ln(x^2 + 1) + 2 \ln \sin x - \frac{1}{2} \ln(x^3 + 2).
\]

Then we take the derivative of both sides: \(\frac{y'}{y} = \frac{2x}{x^2 + 1} + \frac{2 \cos x}{\sin x} - \frac{3x^2}{2(x^3 + 2)}\). Then substituting \(f(x)\) yields

\[
f'(x) = y' = \left( \frac{x^2 + 1}{\sqrt{x^3 + 2}} \right) \left( \frac{2x}{x^2 + 1} + \frac{2 \cos x}{\sin x} - \frac{3x^2}{2(x^3 + 2)} \right).
\]

3. We’ll use the substitution \(u = 1 + e^{3x}\), and so \(du = 3e^{3x} \, dx\). Also, if \(x = 0\) then \(u = 1 + e^0 = 2\) and if \(x = \ln 2\) then \(u = 1 + e^{3\ln 2} = 1 + 8 = 9\). Therefore

\[
\int_0^{\ln 2} e^{3x} \frac{1}{1 + e^{3x}} \, dx = \int_2^9 \frac{1}{3u} \, du = \frac{1}{3} \ln |u| \bigg|_2^9 = \frac{1}{3} (\ln 9 - \ln 2).
\]

4. We use logarithmic differentiation. Letting \(y = (\sin x)^{(x^2+1)}\), we have \(\ln y = (x^2 + 1) \ln(\sin x)\). Differentiating both sides (using the chain rule) yields

\[
\frac{1}{y} \frac{dy}{dx} = 2x \ln(\sin x) + \frac{(x^2 + 1) \cos x}{\sin x}.
\]

Multiply both sides by \(y\) to obtain \(\frac{dy}{dx} = (\sin x)^{(x^2+1)} \left[ 2x \ln(\sin x) + \frac{(x^2 + 1) \cos x}{\sin x} \right]\).

5. Letting \(P(t)\) be the population \(t\) years after Jan 01, 2000, the formula for exponential growth is \(P(t) = P(0)e^{rt}\). Fill in the value \(P(0) = 5000\), and solve for \(r\) using \(P(5) = 5500\).

\[
5500 = 5000e^{5r} \Rightarrow r = \frac{1}{5} \ln \left( \frac{5500}{5000} \right) = \frac{1}{5} \ln(1.1)
\]

Then \(P(t) = 5000e^{\ln(1.1)t/5} = 5000 \cdot 1.1^{t/5}\). Now set \(t = 20\) to obtain the answer. \(P(20) = 5000e^{4\ln(1.1)} = 5000(1.1^4)\).

6. Set \(u = \ln x\). Then \(du = \frac{1}{x} \, dx\), and the limits are \(\ln(1) = 0\) and \(\ln(e^{1/\sqrt{2}}) = 1/\sqrt{2}\). The integral is transformed to

\[
\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-u^2}} \, du
\]

which we recognize immediately to be \(\arcsin u|_0^{1/\sqrt{2}} = \pi/4 - 0 = \pi/4\).
7. As \( x \to \infty \), both \((\ln x)^2\) and \(x \to \infty\), so this is indeterminate of form \(\frac{\infty}{\infty}\). Thus we apply l’Hospital’s Rule:
\[
\lim_{x \to \infty} \frac{(\ln x)^2}{x} = \lim_{x \to \infty} \frac{2\ln x}{1} = \lim_{x \to \infty} \frac{2\ln x}{x}.
\]
But both \(2\ln x\) and \(x \to \infty\) as \(x \to \infty\), so this is still of indeterminate form \(\frac{\infty}{\infty}\). So we apply l’Hospital’s Rule again:
\[
\lim_{x \to \infty} \frac{2\ln x}{x} = \lim_{x \to \infty} \frac{2}{x} = \lim_{x \to \infty} \frac{2}{x} = 0.
\]

8. We use integration by parts, with \(u = x\) and \(dv = \sin(2x)\). Then \(du = dx\) and \(v = -\cos(2x)\). Thus
\[
\int x \sin(2x) \, dx = -\frac{x \cos(2x)}{2} - \int -\cos(2x) \, dx = -\frac{x \cos(2x)}{2} + \frac{\sin(2x)}{4} + C.
\]

9. Our goal is to do a \(u\)-substitution, with \(u = \tan x\). Thus, \(du = \sec^2(x) \, dx\) so we will leave one \(\sec^2 x\) to become the \(du\), and convert the other \(\sec^2 x\) into \(1 + \tan^2 x\). So we have:
\[
\int \tan^{100} x \sec^4 x \, dx = \int \tan^{100} x (1 + \tan^2 x) \sec^2(x) dx = \int u^{100}(1 + u^2) \, du
\]
We integrate, and then back substitute to get
\[
\int \tan^{100} x \sec^4 x \, dx = \int u^{100} + u^{102} \, du = \frac{u^{101}}{101} + \frac{u^{103}}{103} + C = \frac{\tan^{101} x}{101} + \frac{\tan^{103} x}{103} + C.
\]

10. Since \((x - 1)\) is a polynomial of degree 1, then we get the term \(A/(x - 1)\). Secondly, \((x - 5)\) is a polynomial of degree 1, with exponent 2, hence we get the terms \(B/(x - 5)\) and \(C/(x - 5)^2\). Finally, \((x^2 + 1)\) is an irreducible polynomial of degree 2, so we get the term \((Dx + E)/(x^2 + 1)\). So, the correct form of the partial fraction decomposition of \(f(x)\) is
\[
\frac{A}{x - 1} + \frac{B}{x - 5} + \frac{C}{(x - 5)^2} + \frac{Dx + E}{x^2 + 1}.
\]

11. Let \(L = \lim_{x \to 0^+} (\tan(x))^x\). Taking logarithm to both sides of the equation, we get
\[
\ln(L) = \ln \left( \lim_{x \to 0^+} (\tan(x))^x \right) = \ln \lim_{x \to 0^+} ((\tan(x))^x) = \lim_{x \to 0^+} x \ln(\tan(x)) = \lim_{x \to 0^+} \frac{\ln(\tan(x))}{1/x}.
\]
Since this is an indeterminate of the form \(-\infty/\infty\), by L’Hôpital’s rule we get
\[
\ln(L) = \lim_{x \to 0^+} \frac{\sec^2(x)/(\tan(x))}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{\sin(x) \cos(x)}.
\]
This is again an indeterminate of the form \(0/0\), by L’Hôpital’s rule we get
\[
\ln(L) = \lim_{x \to 0^+} \frac{-2x}{\cos^2(x) - \sin^2(x)} = 0.
\]
Since \(\ln(L) = 0\) we get \(L = 1\).
12. Let’s calculate the integral using the partial fraction method. The right form for the partial fraction decomposition of $10/(x - 3)(x^2 + 1)$ is

$$
\frac{10}{(x - 3)(x^2 + 1)} = \frac{A}{x - 3} + \frac{Bx + C}{x^2 + 1}.
$$

Hence, $10 = A(x^2 + 1) + (Bx + C)(x - 3)$. Setting $x = 3$, we get $10 = 10A$, this is, $A = 1$. Setting $x = 0$, we get $10 = 1(1) + C(-3)$, this is, $C = -3$. Setting $x = 1$, we get $10 = (1)(2) + (B - 3)(-2)$, this is, $B = -1$. So

$$
\int \frac{10}{(x - 3)(x^2 + 1)} dx = \int \frac{1}{(x - 3)} + \frac{-x + 3}{(x^2 + 1)} dx = \int \frac{1}{(x - 3)} dx + \int \frac{-x}{(x^2 + 1)} dx + \int \frac{-3}{(x^2 + 1)} dx.
$$

Now, making the substitution $w = x - 3$ on the first integral, we get

$$
\int \frac{1}{(x - 3)} dx = \ln(|x - 3|) + C_1.
$$

For the second integral, let $u = x^2 + 1$, $du = 2xdx$. Hence

$$
\int \frac{-x}{(x^2 + 1)} dx = -\frac{1}{2} \int \frac{1}{u} du = \ln |u| = -\frac{1}{2} \ln(|x^2 + 1|) + C_2.
$$

Finally, for the third integral, we get

$$
\int \frac{-3}{(x^2 + 1)} dx = -3 \int \frac{1}{(x^2 + 1)} dx = -3 \arctan(x) + C_3
$$

This shows,

$$
\int \frac{10}{(x - 3)(x^2 + 1)} dx = \ln(|x - 3|) - \frac{1}{2} \ln(|x^2 + 1|) - 3 \arctan(x) + C.
$$

13. Since the argument of the integral is of the form $\sqrt{x^2 + a^2}$, we make the substitution $x = 3 \tan(\theta)$, $dx = 3 \sec^2(\theta)d\theta$ (and remember to change the limits of integration), which yields,

$$
\int_0^3 \frac{1}{\sqrt{x^2 + 9}} dx = \int_0^{\pi/4} \frac{3 \sec^2(\theta)}{\sqrt{9 \sec^2(\theta) + 9}} d\theta.
$$

We factor the 9 out of the square root and then make the substitution $1 + \tan^2 = \sec^2$ to get

$$
\int_0^3 \frac{1}{\sqrt{x^2 + 9}} dx = \int_0^{\pi/4} \frac{3 \sec^2(\theta)}{3 \sqrt{\sec^2(\theta) + 9}} d\theta = \int_0^{\pi/4} \sec(\theta) d\theta = \ln |\sec(\theta) + \tan(\theta)|_0^{\pi/4},
$$

where in the last equality, we applied the last formula from the ‘list of useful trigonometric formulas’ at the end of the exam. Evaluating, we get

$$
\int_0^3 \frac{1}{\sqrt{x^2 + 9}} dx = \ln |2/\sqrt{2} + 1| - \ln |1 + 0| = \ln |\sqrt{2} + 1|
$$

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