Exam I Solutions
Math 10560, Spring 2014

1. (6 pts) The function

$$f(x) = x^3 + x + \ln(x)$$

is one-to-one (there is no need to check this). What is $(f^{-1})'(2)$?

Solution: By guess and check we notice that $f(1) = 2$ so $f^{-1}(2) = 1$. Furthermore

$$f'(x) = 3x^2 + 1 + \frac{1}{x}$$

and thus

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

2. (6 pts) Find the derivative of the function

$$f(x) = \frac{(x^2-1)^5(x^2+x+1)^2}{\sqrt{x^2+1}}.$$

(Logarithmic differentiation might help.)

Solution: Notice that

$$\ln(f(x)) = 5\ln(x^2-1) + 2\ln(x^2+x+1) - \frac{1}{2}\ln(x^2+1).$$

Differentiating both sides with respect to x we have

$$\frac{f'(x)}{f(x)} = \frac{10x}{x^2-1} + \frac{4x+2}{x^2+x+1} - \frac{x}{x^2+1}.$$

Finally, we solve for $f'(x)$:

$$f'(x) = \frac{(x^2-1)^5(x^2+x+1)^2}{\sqrt{x^2+1}}\left[\frac{10x}{x^2-1} + \frac{4x+2}{x^2+x+1} - \frac{x}{x^2+1}\right].$$

3. (6 pts) Compute the integral

$$\int_0^{\log_3 5} \frac{3^x}{1+3^x}dx.$$

Solution:
Let $u = 1 + 3^x$. Then $du = \ln(3)\cdot 3^x dx$, so

$$\int_0^{\log_3 5} \frac{3^x}{1+3^x}\,dx = \frac{1}{\ln(3)}\int_2^6 \frac{1}{u}\,du = \frac{1}{\ln(3)}\ln|u|\Big|_2^6 = \frac{1}{\ln(3)}(\ln(6) - \ln(2)).$$

4. (6 pts) Compute the integral

$$\int_1^e \frac{1}{x(1+(\ln x)^2)} dx.$$

Solution:
Let $u = \ln(x)$. Then $du = \frac{1}{x} dx$, so

$$\int_1^e \frac{1}{x(1+(\ln\, x)^2)}\, dx = \int_0^1 \frac{1}{1+u^2}\, dx = \arctan(u)\Big|_0^1 = \frac{\pi}{4}.$$

5. (6 pts) Simplify the function

$$\cos(\sin^{-1}\left(\frac{x}{2}\right)).$$

Solution:
Let $\theta = \sin^{-1}\left(\frac{x}{2}\right)$. Then $\sin(\theta) = \frac{x}{2}$ so if we let $x =$ opposite and $2 =$ hypotenuse then we have adjacent is $\sqrt{4-x^2}$, so

$$\cos(\theta) = \cos(\arcsin(\frac{x}{2})) = \frac{\sqrt{4-x^2}}{2}.$$

6. (6 pts) Evaluate the limit

$$\lim_{x\to 0^+} \frac{\tan x}{x^2}.$$

Solution:
Substituting 0 into $\frac{\tan x}{x^2}$, we see that this limit is indeterminate of form $\frac{0}{0}$. We apply l'Hospital's rule to obtain

$$\begin{aligned}\lim_{x\to 0^+} \frac{\tan x}{x^2} &= \lim_{x\to 0^+} \frac{\sec^2 x}{2x} \\ &= \frac{\sec^2(0)}{2\cdot 0} \\ &= \frac{1}{0} \\ &= +\infty.\end{aligned}$$

(Note that $\sec^2(0) = \frac{1}{\cos^2(0)} = \frac{1}{1} = 1$.) The sign is positive because $\sec^2(x) > 0$ for all $0 \le x \le \frac{\pi}{2}$ (in fact, $\sec^2(x) > 0$ wherever it is defined), and $2x > 0$ when $x > 0$.

7. (6 pts) Evaluate the integral

$$\int_0^1 x\, e^{2x}\, dx.$$

Solution:
We use integration by parts with

$$u = x \qquad dv = e^{2x} dx$$

$$du = dx \qquad v = \int e^{2x} dx = \frac{1}{2} e^{2x}.$$

This is a good choice, since $\int v du = \int v dx = \int 1/2 e^{2x} dx$ is easy to integrate.

Note that if instead we had chosen $u = e^{2x}$ and $dv = x dx$, we would then have $du = 2e^{2x} dx, v = x^2/2$, and the integral $\int v du = \int x^2 e^{2x} dx$ would be more complicated than the integral we started with.

Now we solve the original integral:

$$\begin{aligned}\int_0^1 x\, e^{2x}\, dx &= \frac{1}{2} x e^{2x}\Big|_0^1 - \frac{1}{2}\int_0^1 e^{2x} dx \\ &= \left(\frac{1}{2}e^2 - 0\right) - \frac{1}{4}e^{2x}\Big|_0^1 \\ &= \frac{1}{2}e^2 - \left(\frac{1}{4}e^2 - \frac{1}{4}e^0\right) \\ &= \frac{e^2+1}{4}.\end{aligned}$$

8. (6 pts) Evaluate the integral

$$\int \sin(5x)\ \cos(3x)\, dx.$$

Note: One of the formulas given on the last page of the exam may help you with this problem.

Solution:
We use the formula $\sin x \cos y = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$ from the last page of the exam. Then

$$\begin{aligned}\int \sin(5x)\ \cos(3x)\, dx &= \int \frac{1}{2}\left[\sin(2x) + \sin(8x)\right]\ dx \\ &= \frac{1}{2}\left[\frac{-\cos(2x)}{2} + \frac{-\cos(8x)}{8}\right] + C \\ &= -\frac{1}{2}\left[\frac{\cos(2x)}{2} + \frac{\cos(8x)}{8}\right] + C.\end{aligned}$$

9. (6 pts) Evaluate the integral

$$\int_0^{\frac{\pi}{4}} \tan^{100} x \sec^4 x \, dx.$$

Solution: This is an integral of the form $\int \sec^m x \tan^n x \; dx$. Our goal is to use the substitution $u = \tan(x)dx$. Since $du = \sec^2 x dx$ we leave one factor of $\sec^2 x$ and convert the other using $\sec^2 x = 1 + \tan^2 x$. Thus

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \tan^{100} x \sec^4 x \, dx &= \int_0^{\frac{\pi}{4}} \tan^{100} x (1 + \tan^2 x) \sec^2 x \; dx \\
&= \int_{\tan(0)}^{\tan\left(\frac{\pi}{4}\right)} u^{100}(1 + u^2) \; du \\
&= \int_0^1 u^{100} + u^{102} \; du \\
&= \left(\frac{u^{101}}{101} + \frac{u^{103}}{103} \right) \Bigg|_0^1 \\
&= \frac{1}{101} + \frac{1}{103}.
\end{aligned}$$

10. (6 pts) Which of the following expressions gives the correct form of the partial fraction decomposition of the function f shown below?

$$f(x) = \frac{3x^2 + 2x + 1}{(x-1)(x-4)^2(x^2+1)^2}$$

Solution:
The denominator $Q(x) = (x-1)(x-4)^2(x^2+1)^2$ is the product of a linear term, a repeated linear term, and a repeated irreducible quadratic factor.
For the linear factor we need a term of the form $\frac{A}{x-1}$.
For the repeated linear factor we need a term of the form $\frac{B}{x-4} + \frac{C}{(x-4)^2}$.
Finally, for the repeated irreducible quadratic we must add a term of the form

$\frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$.

Combining these results, we see that the partial fraction decomposition of $f(x)$ has form

$$\frac{A}{x-1} + \frac{B}{x-4} + \frac{C}{(x-4)^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}.$$

11. (10 pts) let $M(t)$ denote the amount of a chemical substance remaining after t years where the initial amount is given by $M(0)$. The rate of decay of the substance is such that 40% of the initial amount is left after 10 years. It is known that the substance decreases at a rate proportional to the amount present a time t, that is $M'(t) = kM(t)$ for some constant k.

(a) What is the value of k?

Solution: We are given that $M(t) = M(0)e^{kt}$ and $M(10) = 0.4M(0)$.

Thus $0.4M(0) = M(10) = M(0)e^{k \cdot 10}$, so $0.4 = e^{10k}$.

Taking the natural log of both sides of this last equation yields $\ln(0.4) = 10k$, or

$$k = \ln(0.4)/10.$$

(b)What is the half-life of this substance (what is the amount of time it takes to decay to 50% of its original size)?

Solution: We need to find the time t when $\frac{M(t)}{M(0)} = 0.5$:

$$\begin{aligned}
&\frac{M(t)}{M(0)} = 0.5 \\
&\iff \frac{M(0)e^{kt}}{M(0)} = 0.5 \\
&\iff e^{kt} = 0.5 \\
&\iff kt = \ln(0.5) \\
&\iff t = \ln(0.5)/k \\
&\iff t = \frac{\ln(0.5)}{\ln(0.4)/10} = \frac{10\ln(0.5)}{\ln(0.4)} \approx 7.5\text{years.}
\end{aligned}$$

12.(15 pts) Compute the integral

$$\int \frac{x^2+3x}{(x-2)(x^2+2x+2)}dx \ .$$

Solution: Since $x-2$ is linear and x^2+2x+2 is irreducible, the partial fraction decomposition of the integrand has form

$$\frac{x^2+3x}{(x-2)(x^2+2x+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+2} \qquad (1).$$

Multiplying both sides of equation (1) by $(x-2)(x^2+2x+2)$ yields

$$x^2+3x = (x^2+2x+2)(A) + (x-2)(Bx+C) \qquad (2).$$

If we let $x=2$ in equation (2), we can immediately solve for A:

$$\begin{aligned} 2^2+3\cdot 2 &= (2^2+2\cdot 2+2)A + 0(Bx+C) \\ 10 &= 10\cdot A \\ A &= 1. \end{aligned}$$

Putting $A=1$ into equation (2) and putting like terms together, we have

$$\begin{aligned} x^2+3x &= (x^2+2x+2)(1) + (x-2)(Bx+C) \\ x^2+3x &= (x^2+2x+2) + (Bx^2+Cx-2Bx-2C) \\ 0x^2+1\cdot x-2 &= Bx^2+(C-2B)x-2C \qquad (3). \end{aligned}$$

From (3) we can see that $B=0$, and $-2=-2\cdot C \implies C=1$. Now we are ready to solve the integral:

$$\begin{aligned} \int \frac{x^2+3x}{(x-2)(x^2+2x+2)}dx &= \int (\frac{1}{x-2} + \frac{1}{x^2+2x+2})dx \\ &= \ln|x-2| + \int \frac{dx}{x^2+2x+2} \\ &= \ln|x-2| + \int \frac{dx}{(x+1)^2+1} \\ &= \ln|x-2| + \tan^{-1}(x+1) + C. \end{aligned}$$

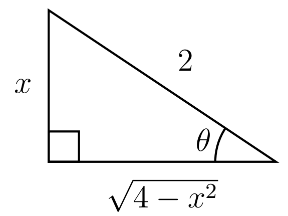
. **13.** (15 pts) Calculate the integral

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx\ .$$

Note: One of the formulas given on the last page of the exam may help you with this problem.

Solution:

The integral involves x and $\sqrt{4-x^2}$, so we draw a right triangle with legs x, $\sqrt{4-x^2}$ to help us determine what trig substitution to use:



We now read the following equations from our picture:

$$2\sin\theta = x$$
$$2\cos\theta = \sqrt{4-x^2}.$$

We are almost ready to perform the substitution, but first we must write dx in terms of $d\theta$. By applying d to both sides of the equation $2\sin\theta = x$, we have

$$2\cos\theta d\theta = dx$$

Finally, note that $\theta = \sin^{-1}(x/2)$, so our original limits of integration (0 and 1) will change to $\sin^{-1}(0) = 0$ and $\sin^{-1}(1/2) = \pi/6$, respectively.

Thus:

$$\begin{aligned}
\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int_0^{\pi/6} \frac{(2\sin(\theta))^2 \cdot 2\cos(\theta)d\theta}{2\cos(\theta)} \\
&= \int_0^{\pi/6} 4\sin^2(\theta)d\theta \qquad (1) \\
&= 4\int_0^{\pi/6} \frac{1}{2}(1-\cos(2\theta))d\theta \qquad (2) \\
&= 2\int_0^{\pi/6} (1-\cos(2\theta)d\theta \\
&= \left[2(\theta - \frac{1}{2}\sin(2\theta))\right]_0^{\pi/6} \\
&= 2\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{2}.
\end{aligned}$$

Note: to get from (1) to (2) we have used the identity $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$.