SOLUTIONS TO EXAM 1, MATH 10560

1. *Simplify the following expression for* x

$$x = \log_3 81 + \log_3 \frac{1}{9} \ .$$

Solution:

$$x = \log_3 81 + \log_3 \frac{1}{9} = \log_3 \frac{81}{9} = \log_3 9 = 2 \ .$$

2. *The function* $f(x) = x^3 + 3x + e^{2x}$ *is one-to-one. Compute* $(f^{-1})'(1)$.

Solution:

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

$f^{-1}(1) = 0$ and $f'(x) = 3x^2 + 3 + 2e^{2x}$. Hence $f'(f^{-1}(1)) = f'(0) = 5$. Therefore $(f^{-1})'(1) = \frac{1}{5}$.

3. *Differentiate the function*

$$f(x) = \frac{(x^2-1)^4}{\sqrt{x^2+1}}.$$

Solution: Use logarithmic differentiation.

$$\ln f = 4\ln(x^2-1) - \frac{1}{2}\ln(x^2+1)$$

$$\frac{f'}{f} = \frac{8x}{x^2-1} - \frac{x}{x^2+1}$$

$$f'(x) = \frac{x(x^2-1)^4}{\sqrt{x^2+1}}\left(\frac{8}{x^2-1} - \frac{1}{x^2+1}\right).$$

4. *Compute the integral*

$$\int_{2e}^{2e^2} \frac{1}{x(\ln \frac{x}{2})^2} dx.$$

Solution: Make the substitution $u = \ln \frac{x}{2}$ with $dx = xdu$.

$$\int_{2e}^{2e^2} \frac{1}{x(\ln \frac{x}{2})^2} dx = \int_1^2 \frac{1}{u^2} du = \left[-\frac{1}{u}\right]_1^2 = \frac{1}{2} \ .$$

5. *Compute the limit*

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}}.$$

Solution:

$$\begin{aligned}\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^{2x}-e^{-2x}} &= \lim_{x\to\infty}\frac{e^x(1-e^{-2x})}{e^{2x}(1-e^{-4x})}\\ &= \lim_{x\to\infty}\frac{1-e^{-2x}}{e^x(1-e^{-4x})} = 0.\end{aligned}$$

6. *Find* $f'(x)$ *if*

$$f(x) = x^{\ln x} .$$

Solution: One method is to use logarithmic differentiation.

$$\ln y = \ln(x^{\ln x}) = (\ln x)(\ln x) = (\ln x)^2.$$

$$\frac{y'}{y} = \frac{2\ln x}{x} .$$

Therefore $f'(x) = y' = 2(\ln x)x^{(\ln x)-1}$.

7. *Calculate the following integral*

$$\int_0^1 \frac{\arctan x}{1+x^2}\, dx .$$

Solution: Make the substitution $u = \arctan x$ with $dx = (1+x^2)du$.

$$\int_0^1 \frac{\arctan x}{1+x^2}\, dx = \int_0^{\frac{\pi}{4}} u\, du = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32} .$$

8. *Evaluate the integral*

$$\int_0^{\pi/2} \sin^3(x)\cos^5(x)dx.$$

Solution: Use the identity $1-\cos^2(x) = \sin^2(x)$.

$$\begin{aligned}\int_0^{\pi/2} \sin^3(x)\cos^5(x)dx &= \int_0^{\pi/2} (1-\cos^2(x))\sin(x)\cos^5(x)dx\\ &= -\int_1^0 u^5 - u^7\, du \quad (u=\cos(x),\ du = -\sin(x)du)\\ &= \int_0^1 u^5-u^7\, du\\ &= \left[\frac{u^6}{6}-\frac{u^8}{8}\right]_0^1\\ &= \frac{1}{6}-\frac{1}{8} = \frac{1}{24}.\end{aligned}$$

9. *Evaluate the limit*

$$\lim_{x\to 0}\left(cosh(x)\right)^{1/x^2}.$$

Solution: The limit has indeterminate form 1^∞. Let $L = \lim_{x\to 0}\left(\cosh(x)\right)^{1/x^2}$.

$$\begin{aligned}\ln L &= \lim_{x\to 0}\ln\left(\left(\cosh(x)\right)^{1/x^2}\right)\\ &= \lim_{x\to 0}\frac{\ln\left(\cosh(x)\right)}{x^2}\\ &= \lim_{x\to 0}\frac{\tanh(x)}{2x}\quad\text{(l'Hospital's rule)}\\ &= \lim_{x\to 0}\frac{\operatorname{sech}^2(x)}{2}\quad\text{(l'Hospital's rule)}\\ &= \frac{1}{2}.\end{aligned}$$

Therefore $L = e^{\frac{1}{2}}$.

10. *Evaluate the integral*

$$\int x^2\cos(2x)dx.$$

Solution:

$$\begin{aligned}&\int x^2\cos(2x)dx\\ =&\frac{1}{2}x^2\sin(2x) - \int x\sin(2x)dx\quad\text{(integration by parts)}\\ =&\frac{1}{2}x^2\sin(2x) - \left[-\frac{1}{2}x\cos(2x) - \int -\frac{1}{2}\cos(2x)dx\right]\quad\text{(integration by parts)}\\ =&\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C\end{aligned}$$

11. *Evaluate*

$$\int \frac{1}{3}x^3\sqrt{9-x^2}\ dx.$$

Solution: Two approaches work: trigonometric substitution with $x = 3\sin\theta$ and u substitution with $u = 9 - x^2$. The method of trigonometric substitution is outlined here, although the latter method may be somewhat easier.

$$\begin{aligned}
\int \frac{1}{3}x^3\sqrt{9-x^2}\,dx &= \int 81\sin^3\theta\cos^2\theta d\theta \quad (x = 3\sin\theta,\ dx = 3\cos\theta du) \\
&= \int 81(1-\cos^2\theta)\sin\theta\cos^2\theta d\theta \quad (u = \cos\theta,\ du = -\sin\theta d\theta) \\
&= \int 81(u^4 - u^2)du \\
&= \frac{81\cos^5\theta}{5} - 27\cos^3\theta + C \quad \left(\cos\theta = \frac{1}{3}\sqrt{9-x^2}\right) \\
&= \frac{(9-x^2)^{\frac{5}{2}}}{15} - (9-x^2)^{\frac{3}{2}} + C.
\end{aligned}$$

12. *Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = kC(t)$, where k is a constant. The initial concentration of the drug is 4 mg/ml. After 5 hours, the concentration is 3 mg/ml.*
(a) Give a formula for the concentration of the drug at time t.
(b) How much drug will there be in 10 hours?
(c) How long will it take for the concentration to drop to 0.5 mg/ml?
Solution: (a)

$$\begin{aligned}
C(t) = C(0)e^{kt} &= 4e^{kt} \\
C(5) = 3 &= 4e^{k5} \quad \text{(solve for k)} \\
k &= \frac{1}{5}\ln\left(\frac{3}{4}\right) \quad \text{(substitute into } C(t)) \\
C(t) &= 4\left(\frac{3}{4}\right)^{\frac{1}{5}t}.
\end{aligned}$$

(b)

$$C(10) = 4\left(\frac{3}{4}\right)^2 = \frac{9}{4}\ .$$

(c)

$$\begin{aligned}
C(t) = 4\left(\frac{3}{4}\right)^{\frac{1}{5}t} &= \frac{1}{2} \quad \text{(solve for t)} \\
t = -5\log_{3/4}(8) &= \frac{-5\ln 8}{\ln 3 - \ln 4}\ .
\end{aligned}$$