SOLUTIONS TO EXAM 2, MATH 10560

1. *Which of the following expressions gives the partial fraction decomposition of the function*

$$f(x) = \frac{x^2 - 2x + 6}{x^3(x-3)(x^2+4)}?$$

Solution: Since x is a linear factor of multiplicity 3, $(x-3)$ is a linear factor of multiplicity 1 and (x^2+4) is an irreducible quadratic factor of multiplicity 1, then

$$\frac{x^2 - 2x + 6}{x^3(x-3)(x^2+4)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-3} + \frac{Ex+F}{x^2+4}.$$

2. *Use the trapezoidal rule with step size $\Delta x = 2$ to approximate the integral $\int_0^4 f(x)dx$.*

Solution: Note

$$n = \frac{4-0}{2} = 2.$$

Then by the trapezoidal rule

$$\int_0^4 f(x)dx \approx \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + f(x_2)) = \frac{2}{2}(2+8+0) = 10.$$

3. *Evaluate the following improper integral:*

$$\int_e^\infty \frac{1}{x(\ln x)^2} dx.$$

Solution: Use the definition of improper integral and make the substitution $u = \ln x$ with $dx = xdu$. Then

$$\int_e^\infty \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_1^{\ln t} \frac{1}{u^2} du$$

$$= \lim_{t\to\infty} \left[-\frac{1}{u} \right]_1^{\ln t} = \lim_{t\to\infty} (-\frac{1}{\ln t} + 1) = 1.$$

4. *Find $\int_{-2}^2 \frac{1}{x+1} dx$.*

Solution: Function $\frac{1}{x+1}$ has an infinite discontinuity at the point $x = -1$. Therefore

$$\int_{-2}^2 \frac{1}{x+1} dx = \int_{-2}^{-1} \frac{1}{x+1} dx + \int_{-1}^2 \frac{1}{x+1} dx,$$

where each of the integrals is improper. Compute the first integral as follows

$$\int_{-2}^{-1} \frac{1}{x+1}dx = \lim_{t\to\ -1} \int_{-2}^{t} \frac{1}{x+1}dx = \lim_{t\to\ -1} [\ln|x+1|]_{-2}^{t} = \lim_{t\to\ -1} \ln|t+1| - \ln 1 = -\infty.$$

Since $\int_{-2}^{-1} \frac{1}{x+1}dx$ diverges, then the initial integral diverges as well.

5. *Which of the following is an expression for the area of the surface formed by rotating the curve $y = 5^x$ between $x = 0$ and $x = 2$ about the y-axis?*

Solution: Distance from the axis of the revolution (y-axis) and the graph of the function $y = 5^x$ is x. Therefore

$$S = \int_a^b 2\pi x\sqrt{1+(y')^2}dx = \int_0^2 2\pi x\sqrt{1+(\ln 5)^2\cdot 25^x}dx.$$

6. *Find the centroid of the region bounded by $y = 1/x$, $y = 0$, $x = 1$, and $x = 2$.*

Solution: Use

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)dx,$$
$$\bar{y} = \frac{1}{2A}\int_a^b f^2(x)dx,$$

where A is the area of the given region. Therefore

$$\bar{x} = \frac{1}{\ln 2}\int_1^2 x\frac{1}{x}dx = \frac{1}{\ln 2}\int_1^2 1dx = \frac{1}{\ln 2}x|_1^2 = \frac{1}{\ln 2}(2-1) = \frac{1}{\ln 2},$$
$$\bar{y} = \frac{1}{2\ln 2}\int_1^2 \frac{1}{x^2}dx = \frac{1}{2\ln 2}[-\frac{1}{x}]|_1^2 = \frac{1}{2\ln 2}(-\frac{1}{2}+1) = \frac{1}{4\ln 2}.$$

7. *Use Euler's method with step size 0.5 to estimate $y(1)$ where $y(x)$ is the solution to the initial value problem*

$$y' = y + 2xy, \qquad y(0) = 1.$$

Solution: Note $\Delta x = 0.5$, $a = 0, b = 1, n = \frac{1-0}{0.5} = 2$. Therefore using $F(x,y) = y + 2xy$ for Euler's method

$$y(0) = y_0 = 1,$$
$$y(0.5) \approx y_1 = y_0 + F(x_0,y_0)\Delta x = y_0 + (y_0 + 2x_0y_0)\Delta x = 1 + (1+0)0.5 = 1.5,$$
$$y(1) \approx y_2 = y_1 + F(x_1,y_1)\Delta x = y_1 + (y_1 + 2x_1y_1)\Delta x = 1.5 + (1.5+1.5)0.5 = 3.$$

8. *Find the solution to the initial value problem*

$$y' = \frac{\sin x}{2y+1}, \quad y(0) = 2.$$

Solution: Separate variables and then integrate

$$(2y+1)y' = \sin x,$$

or

$$\int (2y+1)dy = \int \sin x dx.$$

We get

$$y^2 + y = -\cos x + C.$$

Now use the initial value to find C as follows

$$2^2 + 2 = -1 + C.$$

Hence $C = 7$, and

$$y^2 + y = 7 - \cos x.$$

9. *Find the integral*

$$\int \frac{3x+1}{x^3+x^2} dx.$$

Solution: Use partial fraction decomposition

$$\frac{3x+1}{x^3+x^2} = \frac{3x+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}.$$

Therefore

$$3x+1 = (A+C)x^2 + (A+B)x + B.$$

It follows that

$$A+C=0, \quad A+B=3, \quad B=1,$$
$$A=2, \quad B=1, \quad C=-2,$$

and

$$\int \frac{3x+1}{x^3+x^2} dx = \int (\frac{2}{x} + \frac{1}{x^2} - \frac{2}{x+1}) dx = 2\ln|x| - \frac{1}{x} - 2\ln|x+1| + C.$$

10. *Calculate the integral*

$$\int \frac{dx}{x+\sqrt[3]{x}}.$$

Solution: Make substitution $u = x^{1/3}$. Then $u^3 = x$ and with $dx = 3u^2du$

$$\int \frac{dx}{x+\sqrt[3]{x}} = \int \frac{3u^2du}{u(u^2+1)} = \int \frac{3udu}{u^2+1} = \frac{3}{2}\ln(u^2+1) = \frac{3}{2}\ln(x^{2/3}+1) + C.$$

11. *Calculate the arc length of the curve if* $y = \frac{x^2}{4} - \ln(\sqrt{x})$, *where* $2 \le x \le 4$.
Solution: Recall

$$L = \int_a^b \sqrt{1+(y')^2} dx.$$

Note

$$y' = \frac{x}{2} - \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2}(x - \frac{1}{x}).$$

Thus

$$1+(y')^2 = 1+\frac{1}{4}(x-\frac{1}{x})^2 = 1+\frac{1}{4}(x^2-2x\frac{1}{x}+\frac{1}{x^2}) = 1+\frac{1}{4}(x^2-2+\frac{1}{x^2})$$

$$=1+\frac{1}{4}x^2-\frac{1}{2}+\frac{1}{4x^2} = \frac{1}{4}x^2+\frac{1}{2}+\frac{1}{4x^2} = \frac{1}{4}(x^2+2x\frac{1}{x}+\frac{1}{x^2}) = \frac{1}{4}(x+\frac{1}{x})^2.$$

Therefore

$$L = \int_2^4 \sqrt{1/4(x+1/x)^2}dx = \int_2^4 \frac{1}{2}(x+\frac{1}{x})dx = \frac{1}{2}\left[(\frac{x^2}{2}+\ln x)\right]_2^4 = 3+\frac{1}{2}\ln 2.$$

12. *Solve the initial value problem*

$$xy' + xy + y = e^{-x}$$

$$y(1) = \frac{2}{e}.$$

Solution: This is a linear differential equation. Since it can be reduced to the form

$$y' + (1+\frac{1}{x})y = \frac{e^{-x}}{x},$$

an integrating factor is

$$I(x) = e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x} = xe^x.$$

Multiply both sides of the differential equation by $I(x)$ to get

$$xe^x y' + y(x+1)e^x = 1,$$

and hence

$$(xe^x y)' = 1.$$

Integrate both sides to obtain

$$xe^x y = x + C,$$

or

$$y = e^{-x}(1+\frac{C}{x}).$$

Using the initial value, we have

$$y(1) = \frac{2}{e} = \frac{1}{e}(1+C), \qquad C = 1.$$

Hence

$$y = e^{-x}(1+\frac{1}{x}).$$