1. Calculate \( \lim_{n \to \infty} \frac{(\ln n)^2}{n} \).

**Solution:** Use L’Hospital rule:

\[
\lim_{x \to \infty} \frac{(\ln x)^2}{x} = \lim_{x \to \infty} \frac{2 \ln x}{1} = \lim_{x \to \infty} \frac{2 \ln x}{x} = \lim_{x \to \infty} \frac{2}{1} = 0.
\]

2. Find \( \sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}} \).

**Solution:**

\[
\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4^n}{3 \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4 \cdot 4^{n-1}}{3 \cdot 5^{n-1}} = \frac{4}{3} \sum_{n=1}^{\infty} \left( \frac{4}{5} \right)^{n-1} = \frac{4}{3 \cdot \frac{4}{5} - 1} = 20 \cdot \frac{3}{5} = 12.
\]

3. Discuss the convergence of the series \( \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \).

**Solution:** It converges conditionally. It’s an alternating series and \( b_n = \frac{1}{\sqrt{n}} \). Since \( b_n \) is decreasing and the limit of \( b_n \) is zero, the alternating series converges by the Alternating Series Test. But the series \( \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \) diverges since it’s a \( p \) series and \( p = \frac{1}{2} < 1 \).

4. Use Comparison Tests to determine which one of the following series is divergent.

**Solution:**

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^7 + 1} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^7} \), a \( p \)-series with \( p = \frac{7}{2} > 1 \).

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 8} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), a \( p \)-series with \( p = 2 > 1 \).

(c) \( \sum_{n=1}^{\infty} \frac{n^2 - 1}{n^3 + 100} \) diverges by limit comparison with \( \sum_{n=1}^{\infty} \frac{1}{n} \), a \( p \)-series with \( p = 1 \).

(d) \( \sum_{n=1}^{\infty} 7 \left( \frac{5}{6} \right)^n \) converges since it is a geometric series with \( r = \frac{5}{6} < 1 \).
(e) \( \sum_{n=1}^{\infty} \frac{n}{n+1} \left( \frac{1}{2} \right)^n \) converges by comparison with \( \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \), a geometric series with \( r = \frac{1}{2} < 1 \).

5. Which series below is the MacLaurin series (Taylor series centered at 0) for \( \frac{x^2}{1+x^2} \)?

Solution:
\[
x^2 / (1 + x^2) = x^2 \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n + 2.
\]

6. Find the degree 3 MacLaurin polynomial (Taylor polynomial centered at 0) for the function \( e^{\sqrt{x}} \).

Solution:
\[
e^{\sqrt{x}} = \sum_{n=0}^{\infty} \frac{x^n}{(2n)!}, \quad \frac{1}{1 - x^2} = \sum_{n=0}^{\infty} x^{2n}. \text{ Thus } e^{\sqrt{x}} \cdot \frac{1}{1 - x^2} = (1 + x + \frac{x^3}{6} + \ldots)(1 + x^2 + \ldots) = 1 + x + \frac{3}{2} x^2 + \frac{7}{6} x^3 + \ldots.
\]

7. Which series below is a power series for \( \cos(\sqrt{x}) \)?

Solution: Since \( \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \), we have
\[
\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}.
\]

8. Calculate
\[
\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9}.
\]

Solution: Since \( \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \), we have
\[
\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!},
\]
and
\[
\lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x \to 0} \frac{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots - x^3}{x^9} = \lim_{x \to 0} \frac{-\frac{x^6}{3!} + \cdots}{x^9} = -\frac{1}{6}.
\]
9. Does the series
\[ \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}} \]
converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

**Solution:** Let \( a_n = \frac{(n!)^n}{n^{2n}} \). Since
\[ \lim_{n \to \infty} \frac{n!}{n^2 n!} = \lim_{n \to \infty} \frac{n-1}{n} \cdot (n-2)! = \infty, \]
we have
\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( \frac{n!}{n^2} \right)^n = \infty. \]
Hence \( \lim_{n \to \infty} a_n \neq 0 \). By the Test for Divergence, the series is divergent.

10. Use the Integral Test to discuss whether the series \( \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n} \) converges.

**Solution:** Let \( f(x) = \frac{(\ln x)^2}{x} \). It is continuous and positive when \( x > 1 \). Note
\[ f'(x) = \frac{2 \ln x - (\ln x)^2}{x^2} = \frac{\ln x}{x^2} (2 - \ln x). \]
Then \( f'(x) < 0 \) and hence \( f(x) \) is decreasing for \( x > e^2 \). Therefore, we can use the Integral Test. Next,
\[ \int_{1}^{\infty} \frac{(\ln x)^2}{x} \, dx = \lim_{t \to \infty} \int_{1}^{t} (\ln x)^2 d(\ln x) = \lim_{t \to \infty} (\ln x)^3 \bigg|_{1}^{t} = \lim_{t \to \infty} \frac{(\ln t)^3}{3} = \infty. \]
Hence the improper integral \( \int_{1}^{\infty} \frac{(\ln x)^2}{x} \, dx \) is divergent. By the Integral Test, the series
\[ \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n} \] diverges.

11. Find the radius of convergence and interval of convergence of the power series
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x - 3)^n \]

**Solution:** Set \( a_n = \frac{(-1)^n}{\sqrt{n}} (x - 3)^n \). Using the Ratio Text,
\[ \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n+1}} |x - 3| = |x - 3| < 1. \]
Hence, the radius of convergence is 1, and from \( |x - 3| < 1 \) we get \( 2 < x < 4 \). For the end points, when \( x = 2 \), the series is
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \] which is divergent since it is
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a \( p \)-series with \( p = \frac{1}{2} < 1 \); when \( x = 4 \), the series is \( \sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n}} \) which is convergent since it’s an alternating series, and \( b_n = \frac{1}{\sqrt{n}} \) are decreasing and \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). Hence, the interval of convergence is \( 2 < x \leq 4 \).

12. (a) Show that

\[ \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1 + x^2} \]

provided that \( |x| < 1 \).

(b) Find

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}. \]

**Solution:** (a) Since \( |x| < 1 \), we have \( |x| < 1 \). Hence

\[ \frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \]

(b) Integrate both the left and right hands of (a) to get

\[ \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = \int \frac{1}{1 + x^2} \, dx \]

\[ \Rightarrow \sum_{n=0}^{\infty} \int (-1)^n x^{2n} \, dx = \int \frac{1}{1 + x^2} \, dx \]

\[ \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + C. \]

Letting \( x = 0 \), we have \( C = 0 \). Hence, we have

\[ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x. \]

Let \( x = \frac{1}{\sqrt{3}} \). We get

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}. \]