Math 10560, Practice Exam 3
April 22, 2014

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• No calculators.
• The exam lasts for 1 hour and 15 min.
• Be sure that your name is on every page in case pages become detached.
• Be sure that you have all 11 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

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Multiple Choice  __________
11. __________
12. __________
13. __________
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Multiple Choice

1. (7 pts.) Which statement below is true about the series $\sum_{n=1}^{\infty} \frac{e^n}{n^2 + e^n}$?

\[ \lim_{n \to \infty} \frac{e^n}{n^2 + e^n} = 1 \text{ so the series diverges.} \]

(a) $\lim_{n \to \infty} \frac{e^n}{n^2 + e^n} = 1$ so the series converges.

(b) $\lim_{n \to \infty} \frac{e^n}{n^2 + e^n} = 0$ so the series diverges.

(c) $\lim_{n \to \infty} \frac{e^n}{n^2 + e^n}$ does not exist so the series converges.

(d) $\lim_{n \to \infty} \frac{e^n}{n^2 + e^n} = 0$ so the series converges.

(e) $\lim_{n \to \infty} \frac{e^n}{n^2 + e^n} = 1$ so the series diverges.
2. (7 pts.) The series
\[ \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \]

This series converges conditionally. It’s an alternating series with \( b_n = \frac{1}{\sqrt{n}} \). We have
(i) The sequence \( \{b_n\}_{n=2}^{\infty} \) is decreasing since \( \sqrt{n+1} > \sqrt{n} \) and thus \( b_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = b_n \) for all \( n \geq 2 \).
(ii) \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). Thus the series converges by the Alternating Series Test. But the series
\[ \sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \]
diverges since it’s a \( p \) series and \( p = \frac{1}{2} < 1 \).

(a) diverges even though \( \lim_{n \to \infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 0 \).
(b) does not converge absolutely but does converge conditionally.
(c) diverges because \( \lim_{n \to \infty} \frac{(-1)^{n+1}}{\sqrt{n}} \neq 0 \).
(d) converges absolutely.
(e) diverges because the terms alternate.
3. (7 pts.) Use Comparison Tests to determine which one of the following series is divergent.

(a) \( \sum_{n=1}^{\infty} \frac{1}{3n^2 + 1} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{3n^2} \), a \( p \)-series with \( p = \frac{3}{2} > 1 \).

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^2+8} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), a \( p \)-series with \( p = 2 > 1 \).

(c) \( \sum_{n=1}^{\infty} \frac{n^2-1}{n^3+100} \) diverges by limit comparison with \( \sum_{n=1}^{\infty} \frac{1}{n} \), a \( p \)-series with \( p = 1 \).

(d) \( \sum_{n=1}^{\infty} \frac{n}{n+1} \left(\frac{1}{2}\right)^n \) converges by comparison with \( \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \), a geometric series with \( |r| = \frac{1}{2} < 1 \).

(e) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \) converges by comparison with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), a \( p \)-series with \( p = \frac{3}{2} > 1 \).
4. (7 pts.) Consider the following series

(I) \( \sum_{n=1}^{\infty} \left( \frac{2n^2 + 7}{n^2 + 1} \right)^n \)

(II) \( \sum_{n=2}^{\infty} \frac{2^{1/n}}{n-1} \)

(III) \( \sum_{n=1}^{\infty} \frac{n!}{e^n} \)

For (I), we apply the \( n \)th root test. \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{2n^2 + 7}{n^2 + 1} \)

\( = \lim_{n \to \infty} \frac{2 + \frac{7}{n^2}}{1 + \frac{1}{n^2}} = 2 > 1. \) Therefore the series diverges.

\[ \sum_{n=2}^{\infty} \frac{2^{1/n}}{n-1} \] diverges by direct comparison with the series \( \sum_{n=1}^{\infty} \frac{1}{n} \), since \( \frac{2^{1/n}}{n-1} > \frac{1}{n-1} > \frac{1}{n} \) for all \( n \).

For III, we apply the ratio test, \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{(n+1)!}{e^{n+1}} \frac{e^n}{n!} \)

\( = \lim_{n \to \infty} \frac{n+1}{e} = \infty > 1. \) Therefore the series diverges.

Therefore they all diverge.

Which of the following statements is true?

(a) (I) converges, (II) diverges, and (III) converges.

(b) (I) diverges, (II) diverges, and (III) converges.

(c) (I) converges, (II) diverges, and (III) diverges.

(d) They all diverge.

(e) They all converge.
5. (7 pts.) Which series below conditionally converges?

Recall that a series is conditionally convergent if it is convergent but not absolutely convergent. Note immediately that a) and d) are divergent as their terms tend not to zero as \( n \) goes to infinity. Now, b), c), and e) are convergent by the alternating series test. Further, considering the corresponding series given by taking the absolute value term wise we see that c) and e) are absolutely convergent, while b) is not. Hence b) alone is conditionally convergent.

\[
\begin{align*}
(a) & \quad \sum_{n=1}^{\infty} (-1)^{n-1} \\
(b) & \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \\
(c) & \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^n}{\sqrt{n}} \\
(d) & \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^n}{\sqrt{n}} \\
(e) & \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^n}{n^3}
\end{align*}
\]

6. (7 pts.) Consider the following series

\[
(I) \quad \sum_{n=1}^{\infty} \frac{2^n}{n!} \\
(II) \quad \sum_{n=1}^{\infty} \left( \frac{n^2 + n}{2n^2 + 1} \right)^n
\]

Which of the following statements is true?

Both series converge, to see this we apply the ratio test to series (I) and the root test to series (II). Indeed, let

\[a_n = \frac{2^n}{n!} \quad \text{and} \quad b_n = \left( \frac{n^2 + n}{2n^2 + 1} \right)^n\]

then

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2}{n + 1} = 0 < 1
\]

and

\[
\lim_{n \to \infty} \sqrt[n]{|b_n|} = \lim_{n \to \infty} \frac{n^2 + n}{2n^2 + 1} = \frac{1}{2} < 1
\]

(a) They both converge.
(b) The Ratio Test applied to (I) is inconclusive.
(c) (I) converges and (II) diverges.
(d) (I) diverges and (II) converges.
(e) They both diverge.
7. (7 pts.) Which series below is the MacLaurin series (Taylor series centered at 0) for \( \frac{x^2}{1 + x} \)?

\[
\frac{x^2}{1 + x} = \frac{x^2}{1 - (-x)} = x^2 \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2},
\]

for \( |x| < 1 \).

(a) \( \sum_{n=0}^{\infty} \frac{x^{n+2}}{n + 2} \) 
(b) \( \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n-2}}{n!} \) 
(c) \( \sum_{n=0}^{\infty} (-1)^n x^{2n} \)

(d) \( \sum_{n=0}^{\infty} x^{2n+2} \) 
(e) \( \sum_{n=0}^{\infty} (-1)^n x^{n+2} \)

8. (7 pts.) Which series below is a power series for \( \cos(\sqrt{x}) \)?

**Solution:** Since \( \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \), we have

\[
\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots .
\]

(a) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n + 1)!} \) 
(b) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-\frac{1}{2}}}{(2n)!} \) 
(c) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} \)

(d) \( \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{x}^n}{(2n)!} \) 
(e) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^2 + 1} \)
9. (7 pts.) Calculate
\[ \lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9}. \]

**Hint:** Without MacLaurin series this may be a long problem.

Since \( \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \), we have

\[ \sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots, \]

and

\[ \lim_{x \to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x \to 0} \frac{(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots) - x^3}{x^9} = \lim_{x \to 0} \frac{-\frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots}{x^9} = -\frac{1}{6}. \]

(a) \( \infty \) (b) \( \frac{9}{7} \) (c) 0 (d) \( -\frac{1}{6} \) (e) \( \frac{7}{9} \)

10. (7 pts.) The following is the fifth order Taylor polynomial of the function \( f(x) \) at \( a \)

\[ T_5(x) = 2 - 2(x - a) + \sqrt{5}(x - a)^2 - \frac{\pi}{2}(x - a)^3 + (x - a)^4 + 13(x - a)^5 \]

What is \( f^{(3)}(a) \)?

Looking at the coefficient of \((x - a)^3\), we get

\[ \frac{f^{(3)}(a)}{3!} = -\frac{\pi}{2}. \]

Therefore

\[ f^{(3)}(a) = -3\pi. \]

(a) \( -\frac{\pi}{2} \) (b) \( \sqrt{5} \) (c) \( -3\pi \)

(d) 24 (e) \( 2\sqrt{5} \)
11. (11 pts.) Does the series
\[ \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}} \]
converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

**Remark:** The correct answer with no justification is worth 2 points.

Let \( a_n = \frac{(n!)^n}{n^{2n}} = \left( \frac{n!}{n^2} \right)^n \). Since
\[
\lim_{n \to \infty} \frac{n!}{n^2} = \lim_{n \to \infty} \frac{n-1}{n} \cdot (n-2)! = \infty,
\]
we have
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( \frac{n!}{n^2} \right)^n = \infty.
\]
Hence \( \lim_{n \to \infty} a_n \not= 0 \). By the Test for Divergence, the series is divergent.

Another possibility is to use the Root Test:
\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{n!}{n^2} = \infty.
\]
Since the limit is > 1, the series diverges.
12. (11 pts.) Find the radius of convergence and interval of convergence of the power series

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x - 3)^n \]

**Remark:** The correct answer with no justification is worth 2 points.

Set \( a_n = \frac{(-1)^n}{\sqrt{n}} (x - 3)^n \). Using the Ratio Test,

\[ \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} |x - 3| = |x - 3|. \]

Hence, the radius of convergence is 1, and the series converges absolutely for \(|x - 3| < 1\), or \(2 < x < 4\). For the end points, when \(x = 2\), the series is

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \]

which is divergent since it is a \(p\)-series with \(p = \frac{1}{2} < 1\); when \(x = 4\), the series is

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \]

which is convergent since it’s an alternating series, and \(b_n = \frac{1}{\sqrt{n}}\) is decreasing and

\[ \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0. \] (See the solution to Problem #3 for details.) Hence, the interval of convergence is \(2 < x \leq 4\).
13. (11 pts.)
(a) Show that
\[ \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1 + x^2} \]
provided that \(|x| < 1\).

(b) Find
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}. \]
(Hint: First use term-by-term integration on the series in part (a).)

(a) Since \(|x| < 1\), we have \(|x^2| < 1\). Hence
\[ \frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}. \]

(b) Integrate both the left and right hands of (a) to get
\[ \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = \int \frac{1}{1 + x^2} \, dx \]
\[ \Rightarrow \sum_{n=0}^{\infty} \int (-1)^n x^{2n} \, dx = \int \frac{1}{1 + x^2} \, dx \]
\[ \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + C. \]

Letting \(x = 0\), we have \(C = 0\). Hence, we have
\[ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x. \]

Let \(x = \frac{1}{\sqrt{3}}\). We get
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}. \]
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Multiple Choice _________

11. _________

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