1. Simplify the following expression for $x$

$$x = \log_3 81 + \log_3 \frac{1}{9}.$$ 

Solution:

$$x = \log_3 81 + \log_3 \frac{1}{9} = \log_3 81 \cdot \frac{1}{9} = \log_3 9 = \log_3 3^2 = 2 \log_3 3 = 2.$$ 

2. The function $f(x) = x^3 + 3x + e^{2x}$ is one-to-one. Compute $(f^{-1})'(1)$.

Solution:

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}.$$ 

By trial-and-error we determine that $f^{-1}(1) = 0$. $f'(x) = 3x^2 + 3 + 2e^{2x}$. Hence $f'(f^{-1}(1)) = f'(0) = 5$. Therefore $(f^{-1})'(1) = \frac{1}{5}$.

3. Differentiate the function

$$f(x) = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}}.$$ 

Solution: Use logarithmic differentiation. (Take logarithm of both sides of equation, then do implicit differentiation.)

$$\ln f = 4 \ln(x^2 - 1) - \frac{1}{2} \ln(x^2 + 1)$$

$$\frac{f'}{f} = \frac{8x}{x^2 - 1} - \frac{x}{x^2 + 1}$$

$$f'(x) = \frac{x(x^2 - 1)^4}{\sqrt{x^2 + 1}} \left( \frac{8}{x^2 - 1} - \frac{1}{x^2 + 1} \right).$$

4. Compute the integral

$$\int_{2e}^{2e^2} \frac{1}{x(\ln \frac{x}{2})^2} \, dx.$$ 

Solution: Make the substitution $u = \ln \frac{x}{2}$ with $dx = x \, du$. At $x = 2e$, have $u = 1$ and at $x = 2e^2$ have $u = 2$.

$$\int_{2e}^{2e^2} \frac{1}{x(\ln \frac{x}{2})^2} \, dx = \int_{1}^{2} \frac{1}{u^2} \, du = \left[ -\frac{1}{u} \right]_{1}^{2} = \frac{1}{2}.$$
5. Which of the following expressions gives the partial fraction decomposition of the function

\[ f(x) = \frac{x^2 - 2x + 6}{x^3(x - 3)(x^2 + 4)} \]

**Solution:** Since \( x \) is a linear factor of multiplicity 3, \( (x - 3) \) is a linear factor of multiplicity 1 and \( (x^2 + 4) \) is an irreducible quadratic factor of multiplicity 1, then

\[
\frac{x^2 - 2x + 6}{x^3(x - 3)(x^2 + 4)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x - 3} + \frac{Ex + F}{x^2 + 4}.
\]

6. Find \( f'(x) \) if

\[ f(x) = x^{\ln x} \]

**Solution:** One method is to use logarithmic differentiation. Let \( y = f(x) \).

\[
\ln y = \ln(x^{\ln x}) = (\ln x)(\ln x) = (\ln x)^2.
\]

\[
\frac{y'}{y} = \frac{2\ln x}{x}.
\]

Therefore \( f'(x) = y' = 2(\ln x)x^{(\ln x)-1} \).

7. Calculate the following integral

\[ \int_0^1 \frac{\arctan x}{1 + x^2} \, dx \]

**Solution:** Make the substitution \( u = \arctan x \) with \( dx = (1 + x^2)du \).

\[
\int_0^1 \frac{\arctan x}{1 + x^2} \, dx = \int_0^{\pi/2} u \, du = \left[ \frac{u^2}{2} \right]_0^{\pi/2} = \frac{\pi^2}{32}.
\]

8. Evaluate the integral

\[ \int_0^{\pi/2} \sin^3(x) \cos^5(x) \, dx \]
Solution: Use the identity $1 - \cos^2(x) = \sin^2(x)$.

\[
\int_0^{\pi/2} \sin^3(x) \cos^5(x) \, dx = \int_0^{\pi/2} (1 - \cos^2(x)) \sin(x) \cos^5(x) \, dx
\]

\[
= - \int_1^0 (u^5 - u^7) \, du \quad (u = \cos(x), \, du = -\sin(x) \, dx)
\]

\[
= \int_0^1 (u^5 - u^7) \, du
\]

\[
= \left[ \frac{u^6}{6} - \frac{u^8}{8} \right]_0^1
\]

\[
= \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.
\]

9. Compute the limit

\[
\lim_{x \to 2} \left( \frac{x}{2} \right)^{\frac{1}{x^2}}.
\]

Solution: We have an indeterminate form $1^\infty$. Let $L = \lim_{x \to 2} \left( \frac{x}{2} \right)^{\frac{1}{x^2}}$. Then

\[
\ln L = \lim_{x \to 2} \ln \left( \frac{x}{2} \right)^{\frac{1}{x^2}} = \lim_{x \to 2} \frac{1}{x^2} \ln \left( \frac{x}{2} \right) \quad \text{(l'Hospital's rule)}
\]

\[
= \lim_{x \to 2} \frac{1}{x^2} \cdot \frac{1}{x - 2}
\]

Therefore $L = e^{\frac{1}{2}} = \sqrt{e}$.

10. Evaluate the integral

\[
\int x^2 \cos(2x) \, dx.
\]

Solution:

\[
\int x^2 \cos(2x) \, dx
\]

\[
= \frac{1}{2} x^2 \sin(2x) - \int x \sin(2x) \, dx \quad \text{(integration by parts,}
\]

\[
\text{with } u = x^2 \text{ and } dv = \cos(2x) \, dx, \text{ so } du = 2x \, dx \text{ and } v = \frac{1}{2} \sin(2x)
\]

\[
= \frac{1}{2} x^2 \sin(2x) - \left[ \frac{1}{2} x \cos(2x) - \int \frac{1}{2} \cos(2x) \, dx \right] \quad \text{(integration by parts again)}
\]

\[
= \frac{1}{2} x^2 \sin(2x) + \frac{1}{2} x \cos(2x) - \frac{1}{4} \sin(2x) + C
\]

11. Evaluate

\[
\int \frac{1}{3} x^3 \sqrt{9 - x^2} \, dx.
\]
Solution: Two approaches work: trigonometric substitution with $x = 3 \sin \theta$ and $u$ substitution with $u = 9 - x^2$. The method of trigonometric substitution is outlined here, although the latter method may be somewhat easier.

\[
\int \frac{1}{3} x^2 \sqrt{9 - x^2} \, dx = \int 81 \sin^3 \theta \cos^2 \theta \, d\theta \quad (x = 3 \sin \theta, \ dx = 3 \cos \theta \, d\theta)
\]

\[
= \int 81(1 - \cos^2 \theta) \sin \theta \cos^2 \theta \, d\theta \quad (u = \cos \theta, \ du = -\sin \theta \, d\theta)
\]

\[
= \int 81(u^4 - u^2) \, du
\]

\[
= \frac{81 \cos^5 \theta}{5} - 27 \cos^3 \theta + C \quad (\cos \theta = \frac{1}{3} \sqrt{9 - x^2})
\]

\[
= \frac{(9 - x^2)^{\frac{5}{2}}}{15} - (9 - x^2)^{\frac{3}{2}} + C.
\]

12. Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = kC(t)$, where $k$ is a constant. The initial concentration of the drug is 4 mg/ml. After 5 hours, the concentration is 3 mg/ml.

(a) Give a formula for the concentration of the drug at time $t$.

(b) How much drug will there be in 10 hours?

(c) How long will it take for the concentration to drop to 0.5 mg/ml?

Solution: (a)

\[C(t) = C(0)e^{kt} = 4e^{kt}\]

\[C(5) = 3 = 4e^{k5} \quad \text{(solve for k)}\]

\[k = \frac{1}{5} \ln \left( \frac{3}{4} \right) \quad \text{(substitute into } C(t))\]

\[C(t) = 4 \left( \frac{3}{4} \right)^{\frac{1}{5}t}.\]

(b)

\[C(10) = 4 \left( \frac{3}{4} \right)^{\frac{1}{5} \times 10} = 9/4.\]

(c)

\[C(t) = 4 \left( \frac{3}{4} \right)^{\frac{1}{5}t} = \frac{1}{2} \quad \text{(solve for t)}\]

\[t = -5 \log_{3/4}(8) = \frac{-5 \ln 8}{\ln 3 - \ln 4}.\]