Name: ______________________________

Instructor: __________________________

Math 10560, Practice Exam 2.
March 21, 2012

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 13 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

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5.	(a)	(b)	(c)	(d)	(e)
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Multiple Choice	________
11.	________
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Multiple Choice

1.(6 pts.) Evaluate the improper integral

$$\int_4^\infty \frac{1}{(x-2)(x-3)}\,dx.$$

Using a partial fraction expansion $\int \frac{1}{(x-2)(x-3)}dx = \ln|\frac{x-3}{x-2}| + C.$

Therefore $\int_4^\infty \frac{1}{(x-2)(x-3)}dx = \lim_{t\to\infty} \ln|\frac{t-3}{t-2}| - \ln|\frac{1}{2}| = 0 + \ln 2.$

(a) $\ln 3$ (b) $\ln \frac{1}{2}$ (c) $\ln 2$

(d) the integral diverges (e) $3\ln 2$

2.(6 pts.) What can be said about the integrals

$$(i) \int_0^1 \frac{e^x}{x^2}\,dx;$$

$$(ii) \int_1^\infty \frac{\cos^2 x}{x^2}\,dx?$$

Integral (i) diverges by the Comparison Theorem since the integrand is greater than $\frac{1}{x^2}$.

Integral (ii) converges by the Comparison Theorem since the integrand is less than $\frac{1}{x^2}$.

(a) both (i) and (ii) converge

(b) both (i) and (ii) diverge

(c) (i) converges and (ii) diverges

(d) (i) diverges and (ii) converges

(e) neither integral (i) nor (ii) is improper

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3.(6 pts.) Find the centroid of the triangle with vertices $(-1, 0)$, $(1, 0)$ and $(0, 3)$.

The triangle has area $\frac{1}{2}(2)(3) = 3$ and the moment about the x-axis is

$$M_x = 2\int_0^1 \frac{1}{2}(3-3x)^2 dx = \int_0^1 (9 - 18x + 9x^2)dx = 9 - 9 + 3 = 3.$$

Therefore $\overline{y} = 1$. Using symmetry we see that $\overline{x} = 0$.

(a) $(0, \frac{4}{3})$

(b) $(0, \frac{3}{2})$

(c) $(0, 1)$

(d) $(0, 2)$

(e) $(0, 3)$

4.(6 pts.) Use Euler's method with step size 0.1 to estimate $y(1.2)$ where $y(x)$ is the solution to the initial value problem

$$y' = xy + 1 \qquad y(1) = 0.$$

$$x_0 = 1, \quad y_0 = 0$$
$$x_1 = x_0 + h = 1.1, \quad y_1 = y_0 + h(x_0y_0 + 1) = 0 + (0.1)(1 \cdot 0 + 1) = 0.1$$
$$x_2 = x_1 + h = 1.2, \quad y_2 = y_1 + h(x_1y_1 + 1) = 0.1 + (0.1)((1.1) \cdot (0.1) + 1)$$
$$= 0.1 + 0.1(0.11 + 1) = 0.1 + 0.1(1.11) = 0.1 + 0.111 = 0.211$$

(a) $y(1.2) \approx .201$

(b) $y(1.2) \approx .112$

(c) $y(1.2) \approx .111$

(d) $y(1.2) \approx .101$

(e) $y(1.2) \approx .211$

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5.(6 pts.) Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1+x^2}$$

with initial condition $y(0) = 0$.

Separating variables here gives $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{1+x^2}$.
Solving this gives $\arcsin y = \arctan x + C$ and substituting $y(0) = 0$ we find $C = 0$.
Therefore $y = \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$.

(a) $y = \frac{x^2}{\sqrt{1+x^2}}$

(b) $y = \frac{x}{\sqrt{1+x^2}}$

(c) $y = \frac{x}{1+x}$

(d) $y = \frac{x}{1+x^2}$

(e) $y = \frac{1}{\sqrt{1+x^2}}$

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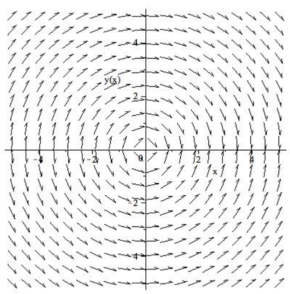
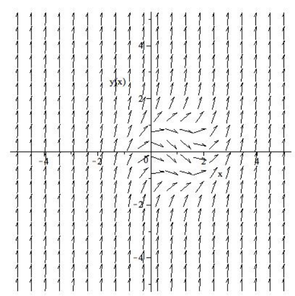
6.(6 pts.) Which of the following gives the direction field for the differential equation

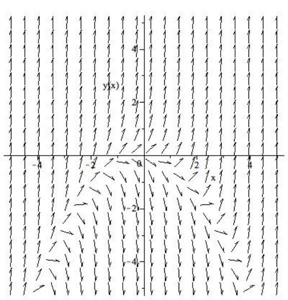
$$y' = y^2 - x^2$$

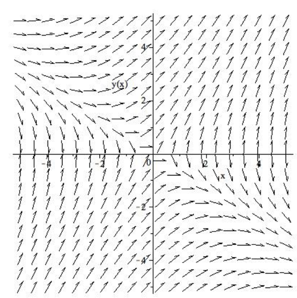
Note the letter corresponding to each graph is at the lower left of the graph.

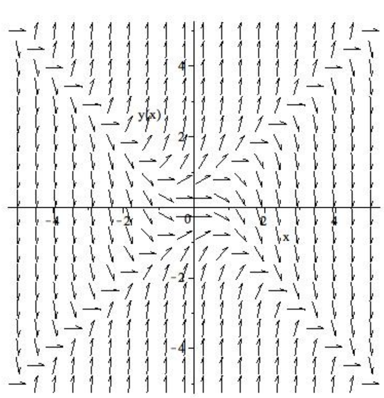
For points on the line $y = x$, we must have $y' = 0$. Also for points on the line $y = -x$, we must have $y' = 0$. Hence along both diagonals f the plane, we must have $y' = 0$ and the answer must be (e).

(a) 

(b) 

(c) 

(d) 

(e) 

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7.(6 pts.) Find a general solution, valid for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, of the differential equation

$$\frac{dy}{dx} - (\tan x)y = 1.$$

The integrating factor here is $I = e^{\int -\tan x dx} = e^{\ln(\cos x)} = \cos x$ and so a general solution is given by $y \cos x = \int \cos x dx = \sin x + C$. Therefore $y = \dfrac{\sin x + C}{\cos x}$.

(a) $y = \dfrac{x + \sin x + C}{\sin x}$ (b) $y = \dfrac{\sin x + C}{\cos x}$ (c) $y = \dfrac{x + \sin x + C}{\cos x}$

(d) $y = \dfrac{\cos x + C}{\sin x}$ (e) $y = \tan x + \cos x + C$

8.(6 pts.) Solve the differential equation

$$8\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + y = 0.$$

We look at the roots of the auxiliary equation $8r^2 - 4r + 1 = 0$.

We get $r = \dfrac{4 \pm \sqrt{16 - 32}}{16} = \dfrac{4 \pm 4i}{16} = \dfrac{1 \pm i}{4} = \dfrac{1}{4} \pm i\dfrac{1}{4}$.

Therefore the solution is of the form

$$e^{\frac{t}{4}}\left[c_1 \cos\left(\frac{t}{4}\right) + c_2 \sin\left(\frac{t}{4}\right)\right]$$

(a) $y(t) = c_1 e^{0.25t} + c_2 e^{-0.25t}$ (b) $y(t) = (c_1 \cos(\frac{1}{4}t) + c_2 \sin(\frac{1}{4}t))$

(c) $y(t) = e^{0.25t}(c_1 \cos(\frac{1}{4}t) + c_2 \sin(\frac{1}{4}t))$ (d) $y(t) = e^t(c_1 \cos(t) + c_2 \sin(t))$

(e) $y(t) = c_1 e^{0.25t} + c_2 t e^{0.25t}$

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9.(6 pts.) Determine which of the following statements are true for the following differential equation:

$$y'' - y' - 6y = 3x.$$

I: $y_c = c_1 e^{3x} + c_2 e^{-2x}$ is the solution to the corresponding homogeneous equation.
II: $y_c = c_1 e^{6x} + c_2 e^{-x}$ is the solution to the corresponding homogeneous equation.
III: $y_p = -\frac{x}{2} + \frac{1}{2}$ is a particular solution.
IV: $y_p = \frac{x+1}{2}$ is a particular solution.

The corresponding homogeneous equation is $y'' - y' - 6y = 0$.
The auxiliary equation $r^2 - r - 6 = 0 \to (r-3)(r+2) = 0$ has roots $r = 3, -2$.
Therefore the general solution to the corresponding homogeneous equation is given by $y_c = c_1 e^{3x} + c_2 e^{-2x}$, where c_1 and c_2 are constants.
Therefore statement I is true and statement II is not.
It is reasonable to expect that there is a particular solution of the form $y_p = Ax + B$. If $y_p = Ax + B$, then $y_p' = A$ and $y_p'' = 0$.
$y_p'' - y_p' - 6y_p = 0 + A - 6[Ax + B] = 3x$. We see that $A = -\frac{1}{2}$ and $B = \frac{1}{12}$. Therefore $y_p = -\frac{1}{2}x + \frac{1}{12}$.
therefore statements III and IV are false.

(a) I and IV

(b) II and IV

(c) I and III

(d) none of the above

(e) II and III

10.(6 pts.) Find the limit: $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+n+1}$

$$\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = \lim_{n\to\infty} \frac{1}{1+1/n+1/n^2} = 1 \neq 0$$

Therefore

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+n+1} \quad \text{does not exist.}$$

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(a) -1

(b) 0

(c) 2

(d) Does not exist

(e) 1

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Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(10 pts.) Solve the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

The auxiliary equation is $r^2 + r - 2 = 0 \to (r+2)(r-1) = 0$.
We have distinct real roots $r_1 = -2$ and $r_2 = 1$,
therefore the general solution to the differential equation is of the form

$$y = c_1 e^{-2x} + c_2 e^x.$$

From the initial condition $y(0) = 1$, we get $c_1 + c_2 = 1$ or $c_1 = 1 - c_2$
$y'(x) = -2c_1 e^{-2x} + c_2 e^x$.
From the initial condition $y'(0) = 2$, we get $-2c_1 + c_2 = 2$
Substituting $1 - c_2$ for c_1, we get $-2(1 - c_2) + c_2 = 2$ or $c_2 = \frac{4}{3}$.
This gives $c_1 = \frac{-1}{3}$.
Therefore

$$y = \frac{-1}{3} e^{-2x} + \frac{4}{3} e^x.$$

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12.(10 pts.) Calculate the arc length of the curve if $y = \frac{x^2}{4} - \ln(\sqrt{x})$, where $2 \leq x \leq 4$.

Solution: Recall

$$L = \int_a^b \sqrt{1 + (y')^2} dx.$$

Note

$$y' = \frac{x}{2} - \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2}\Big(x - \frac{1}{x}\Big).$$

Thus

$$1 + (y')^2 = 1 + \frac{1}{4}\Big(x - \frac{1}{x}\Big)^2 = 1 + \frac{1}{4}\Big(x^2 - 2x\frac{1}{x} + \frac{1}{x^2}\Big) = 1 + \frac{1}{4}\Big(x^2 - 2 + \frac{1}{x^2}\Big)$$

$$= 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}\Big(x^2 + 2x\frac{1}{x} + \frac{1}{x^2}\Big) = \frac{1}{4}\Big(x + \frac{1}{x}\Big)^2.$$

Therefore

$$L = \int_2^4 \sqrt{1/4(x + 1/x)^2} dx = \int_2^4 \frac{1}{2}\Big(x + \frac{1}{x}\Big) dx = \frac{1}{2}\left[\frac{x^2}{2} + \ln x\right]_2^4 = 3 + \frac{1}{2}\ln 2.$$

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13.(10 pts.) (a) Circle the letter below alongside the trapezoidal approximation to

$$\ln 3 = \int_1^3 \frac{1}{x}dx \quad \text{using} \quad n = 8$$

$\boxed{A}$ $$\int_1^3 \frac{1}{x}dx \approx \frac{1}{8}\Big[1+2\Big(\frac{4}{5}\Big)+2\Big(\frac{2}{3}\Big)+2\Big(\frac{4}{7}\Big)+2\Big(\frac{1}{2}\Big)+2\Big(\frac{4}{9}\Big)+2\Big(\frac{2}{5}\Big)+2\Big(\frac{4}{11}\Big)+\Big(\frac{1}{3}\Big)\Big]$$

B $$\int_1^3 \frac{1}{x}dx \approx \frac{1}{12}\Big[1+4\Big(\frac{4}{5}\Big)+2\Big(\frac{2}{3}\Big)+4\Big(\frac{4}{7}\Big)+2\Big(\frac{1}{2}\Big)+4\Big(\frac{4}{9}\Big)+2\Big(\frac{2}{5}\Big)+4\Big(\frac{4}{11}\Big)+\Big(\frac{1}{3}\Big)\Big]$$

C $$\int_1^3 \frac{1}{x}dx \approx \frac{1}{8}\Big[1+\Big(\frac{4}{5}\Big)+\Big(\frac{2}{3}\Big)+\Big(\frac{4}{7}\Big)+\Big(\frac{1}{2}\Big)+\Big(\frac{4}{9}\Big)+\Big(\frac{2}{5}\Big)+\Big(\frac{4}{11}\Big)+\Big(\frac{1}{3}\Big)\Big]$$

(b) Recall that the error E_T in the trapezoidal rule for approximating $\int_a^b f(x)dx$ satisfies

$$\Big|\int_a^b f(x)dx - T_n\Big| = |E_T| \le \frac{K(b-a)^3}{12n^2}$$

whenever $|f''(x)| \le K$ for all $a \le x \le b$.

Use the above error bound to determine a value of n for which the trapezoidal approximation to $\ln 3 = \displaystyle\int_1^3 \frac{1}{x}dx$ has an error

$$|E_T| \le \frac{1}{3}10^{-4}.$$

$$f(x) = \frac{1}{x}, \qquad f'(x) = \frac{-1}{x^2}, \qquad f''(x) = \frac{2}{x^3}$$

Since $|f''(x)| = \dfrac{2}{x^3}$ is decreasing on the interval $1 \le x \le 2$, we have $|f''(x)| \le f''(1) = 2$ for $1 \le x \le 2$. Hence, we can use $K = 2$ in the error bound above.

For the trapezoidal approximation T_n, we have

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{2(3-1)^3}{12n^2} = \frac{16}{12n^2} = \frac{4}{3n^2}$$

If we find a value of n for which $\dfrac{1}{3}10^{-4} \ge \dfrac{4}{3n^2}$, then we will have $|E_T| \le \dfrac{1}{3}10^{-4}$.

$$\frac{1}{3}10^{-4} \ge \frac{4}{3n^2} \quad \rightarrow \quad n^2 \ge 4 \cdot 10^4 \quad \rightarrow \quad n \ge 2 \cdot 10^2 = 200$$

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14.(10 pts.) Find the family of orthogonal trajectories to the family of curves given by

$$y = kx^2.$$

$$\frac{dy}{dx} = 2kx$$

For the family of curves given above

$$y = kx^2 \quad \text{giving} \quad k = \frac{y}{x^2}$$

Thus this family of curves satisfy the differential equation

$$\frac{dy}{dx} = 2\frac{y}{x^2}x = 2\frac{y}{x}.$$

Now using the fact that the product of the derivatives of two orthogonal curves meeting at a point must equal -1, we get that the orthogonal trajectories satisfy the differential equation

$$\frac{dy}{dx} = \frac{-x}{2y}.$$

Separating the variables, we get

$$2ydy = -xdx$$

and

$$2\int ydy = -\int xdx, \quad \text{or} \quad y^2 = \frac{-x^2}{2} + C.$$

Hence our family of orthogonal trajectories is a family of curves of the form

$$y^2 + \frac{x^2}{2} = C,$$

a family of ellipses.

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The following is the list of useful trigonometric formulas:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x - y) + \sin(x + y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x - y) - \cos(x + y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x - y) + \cos(x + y)\big)$$

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