SOLUTIONS TO PRACTICE EXAM 3, MATH 10560

1. *Find* $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}}$.

Solution:

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{3 \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4^n}{3 \cdot 5^{n-1}} = \sum_{n=1}^{\infty} \frac{4 \cdot 4^{n-1}}{3 \cdot 5^{n-1}} = \frac{4}{3} \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^{n-1} = \frac{4}{3} \left(\frac{1}{1 - \frac{4}{5}}\right) = \frac{20}{3}.$$

(The series is geometric with $a = \frac{4}{3}$ and $r = \frac{4}{5}$.)

2. *Discuss the convergence of the series*

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}.$$

Solution: It converges conditionally. It's an alternating series with $b_n = 1/\sqrt{n}$. We have (i) The sequence $\{b_n\}_{n=2}^{\infty}$ is decreasing since $\sqrt{n+1} > \sqrt{n}$ and thus $b_{n+1} = 1/\sqrt{n+1} < 1/\sqrt{n} = b_n$ for all $n \geq 2$. (ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/\sqrt{n} = 0$. Thus the series converges by the Alternating Series Test. But the series $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^{n+1}}{\sqrt{n}}\right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it's a p series and $p = \frac{1}{2} < 1$.

3. *Use Comparison Tests to determine which* **one** *of the following series is divergent.*

Solution: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}} + 1}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, a p-series with $p = \dfrac{3}{2} > 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 8}$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p-series with $p = 2 > 1$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^3 + 100}$ diverges by limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, a p-series with $p = 1$.

(d) $\displaystyle\sum_{n=1}^{\infty} 7\left(\frac{5}{6}\right)^n$ converges since it is a geometric series with $|r| = \dfrac{5}{6} < 1$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1} \left(\frac{1}{2}\right)^n$ converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, a geometric series with $|r| = \dfrac{1}{2} < 1$.

4. Consider the following series

$$(I)\quad \sum_{n=1}^{\infty}\left(\frac{2n^2+7}{n^2+1}\right)^n \qquad (II)\quad \sum_{n=2}^{\infty}\frac{2^{1/n}}{n-1} \qquad (III)\quad \sum_{n=1}^{\infty}\frac{n!}{e^n}$$

Which of the following statements is true?
(I) converges, (II) diverges, and (III) diverges.
They all converge.
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(I) diverges, (II) diverges, and (III) converges.

For (I), we apply the n th root test. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{2n^2+7}{n^2+1}$
$= \lim_{n\to\infty}\frac{2+7/n^2}{1+1/n^2} = 2 > 1$. Therefore the series diverges.

$\sum_{n=2}^{\infty}\frac{2^{1/n}}{n-1}$ diverges by direct comparison with the series $\sum\frac{1}{n}$, since $\frac{2^{1/n}}{n-1} > \frac{1}{n-1} > \frac{1}{n}$ for all n.

For III, we apply the ratio test, $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(n+1)!}{e^{n+1}}\Big/\frac{n!}{e^n}$
$= \lim_{n\to\infty}\frac{n+1}{e} = \infty > 1$. Therefore the series diverges.

Therefore they all diverge.

5. *Which series below is the MacLaurin series (Taylor series centered at* 0*) for* $\frac{x^2}{1+x}$*?*
Solution:

$$\frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^{n+2},$$

for $|x| < 1$.

6. *Which series below is a power series for* $\cos(\sqrt{x})$ *?*

Solution: Since $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, we have

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots .$$

7. *Calculate*

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9}.$$

Solution: Since $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, we have

$$\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots ,$$

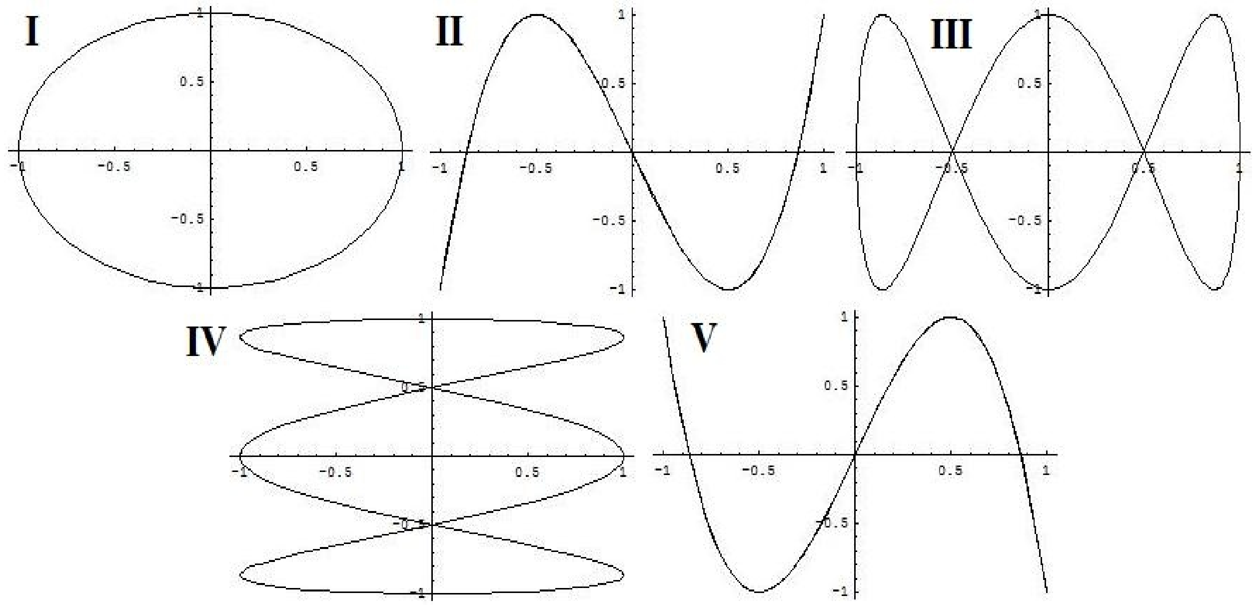
and

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots) - x^3}{x^9} = \lim_{x\to 0} \frac{-\frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots}{x^9} = -\frac{1}{6}.$$

8. Which of the following is a graph of the parametric curve defined by

$$x = \sin(t) \qquad y = \cos(3t)$$

for $0 \le t \le 2\pi$?



We check the values of x and y at some specific values of t: At $t = 0$, $(x, y) = (0, 1)$.
At $t = \frac{\pi}{6}$, $(x, y) = (\frac{1}{2}, 0)$.
At $t = \frac{\pi}{4}$, $(x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.
At $t = \frac{\pi}{3}$, $(x, y) = (\frac{\sqrt{3}}{2}, -1)$.
At $t = \frac{\pi}{2}$, $(x, y) = (1, 0)$. Joining the dots, we see that the answer has to be III.

9. *Does the series*

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$$

converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

Solution: Let $a_n = \frac{(n!)^n}{n^{2n}} = (\frac{n!}{n^2})^n$. Since

$$\lim_{n\to\infty} \frac{n!}{n^2} = \lim_{n\to\infty} \frac{n-1}{n} \cdot (n-2)! = \infty,$$

we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n!}{n^2}\right)^n = \infty.$$

Hence $\lim\limits_{n\to\infty} a_n \neq 0$. By the Test for Divergence, the series is divergent.

Another possibility is to use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty.$$

Since the limit is > 1, the series diverges.

10. *Use the Integral Test to discuss whether the series* $\sum\limits_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ *converges.*

Solution: Let $f(x) = \frac{(\ln x)^2}{x}$. It is continuous and positive when $x > 1$. Note

$$f'(x) = \frac{2x \ln x \cdot \frac{1}{x} - (\ln x)^2}{x^2} = \frac{2\ln x - (\ln x)^2}{x^2} = \frac{\ln x}{x^2}(2 - \ln x),$$

and $2 - \ln x < 0$, i.e., $2 < \ln x$, if $x > e^2$. Then $f'(x) < 0$ and hence $f(x)$ is decreasing for $x > e^2$. Therefore, we can use the Integral Test. We compute the indefinite integral $\int \frac{(\ln x)^2}{x} dx$ using the substitution $u = \ln x$, getting

$$\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C.$$

So

$$\int_1^{\infty} \frac{(\ln x)^2}{x} dx = \lim_{t\to\infty} \int_1^t \frac{(\ln x)^2}{x} dx = \lim_{t\to\infty} \frac{(\ln x)^3}{3}\bigg|_1^t = \lim_{t\to\infty} \frac{(\ln t)^3}{3} = \infty.$$

Hence the improper integral $\int_1^{\infty} \frac{(\ln x)^2}{x} dx$ is divergent. By the Integral Test, the series $\sum\limits_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ diverges. (Note: It is easier to prove this series diverges using the Comparison Test, comparing to the harmonic series. But we were not at liberty to use this test.)

11. *Find the radius of convergence and interval of convergence of the power series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\,(x-3)^n$$

Solution: Set $a_n = \frac{(-1)^n}{\sqrt{n}}(x-3)^n$. Using the Ratio Text,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}|x-3| = |x-3|.$$

Hence, the radius of convergence is 1, and the series converges absolutely for $|x-3| < 1$, or $2 < x < 4$. For the end points, when $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent since it is a p-series with $p = \frac{1}{2} < 1$; when $x = 4$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n}}$ which is convergent since it's an alternating series, and $b_n = \frac{1}{\sqrt{n}}$ is decreasing and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. (See the solution to Problem #3 for details.) Hence, the interval of convergence is $2 < x \leq 4$.

12. *(a) Show that*

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$$

provided that $|x| < 1$.

(b) Find

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}.$$

Solution: (a) Since $|x| < 1$, we have $|x^2| < 1$. Hence

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

(b) Integrate both the left and right hands of (a) to get

$$\int \sum_{n=0}^{\infty}(-1)^n x^{2n} dx = \int \frac{1}{1+x^2}dx$$

$$\Rightarrow \sum_{n=0}^{\infty}\int (-1)^n x^{2n} dx = \int \frac{1}{1+x^2}dx$$

$$\Rightarrow \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + C.$$

Letting $x = 0$, we have $C = 0$. Hence, we have

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x.$$

Let $x = \frac{1}{\sqrt{3}}$. We get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$