Worksheet 10, Math 10560

1. Find a power series representation for the function

\[ \frac{x^2}{1 + x} \]

in the interval \((-1, 1)\).

Solution:

Recall from our knowledge of geometric series that

\[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad \text{whenever } |x| < 1. \quad (1) \]

Replacing \(x\) by \(-x\), we get

\[ \frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} (-x)^n \]

\[ = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{whenever } |x| < 1. \]

Multiplying through by \(x^2\), we obtain

\[ \frac{x^2}{1 + x} = x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2} \quad \text{whenever } |x| < 1. \]
2. Find a power series representation for the function

\[ \frac{x^2}{(1 - x^3)^2} \]

in the interval \((-1, 1)\). Hint: Differentiation of a power series may help.

**Solution:**
Let \( f(x) = \frac{x^2}{(1 - x^3)^2} \), \( g(x) = \frac{1}{1-x^3} \). Then \( g'(x) = \frac{3x^2}{(1-x^3)^2} = 3f(x) \). We can get the desired power series by finding a power series for \( g(x) \), performing term by term differentiation, and then multiplying by \( \frac{1}{3} \). Replacing \( x \) by \( x^3 \) in the power series expansion of \( \frac{1}{1-x} \) (given as equation 1 in the previous problem), we obtain

\[
\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n \quad \text{whenever} \ |x^3| < 1.
\]

\[
= \sum_{n=0}^{\infty} x^{3n} \quad \text{whenever} \ |x| < 1.
\]

Differentiating, we obtain

\[ \frac{d}{dx} \left[ \frac{1}{1-x^3} \right] = \frac{d}{dx} \left[ \sum_{n=0}^{\infty} x^{3n} \right] \]

\[ \Rightarrow \frac{3x^2}{(1-x^3)^2} = \sum_{n=1}^{\infty} 3nx^{3n-1} \]

\[ \Rightarrow \frac{x^2}{(1-x^3)^2} = \sum_{n=1}^{\infty} nx^{3n-1} \quad \text{whenever} \ |x| < 1. \]
3. a) Write down the Taylor series expansion for \( f(x) = \arctan(x) \) about \( x = 0 \).

**Solution:** This well-known Taylor Series was discussed in class:

\[
\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} \quad \text{for } -1 \leq x \leq 1.
\]

b) Compute the following sum. Hint: Use part (a).

\[
\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n + 1}
\]

**Solution:** We can rewrite this series as

\[
\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n + 1} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1}.
\]

The trick is to notice that \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} \) is exactly the series above, with \( x = 1 \) plugged in. So

\[
4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} = 4 \arctan(1)
\]

\[
= 4 \cdot \frac{\pi}{4}
\]

\[
= \pi.
\]

Cool Fact: This allows us to write \( \pi \) as an infinite sum of rational numbers.

\[
\pi = 4 \left[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots + \frac{(-1)^n}{2n + 1} + \cdots \right]
\]

\[
= 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} + \cdots + \frac{4(-1)^n}{2n + 1} + \cdots
\]
4. Find the radius of convergence and interval of convergence of the following power series. Remember to check the endpoints of your interval.

\[
\sum_{n=1}^{\infty} \frac{(-1)^n(4x-1)^n}{2^n\sqrt{n+1}}
\]

**Solution:** We use the ratio test with \( a_n = \frac{(-1)^n(4x-1)^n}{2^n\sqrt{n+1}} \).

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}(4x-1)^{n+1}}{2^{n+1}\sqrt{n+2}} \cdot \frac{2^n\sqrt{n+1}}{(-1)^n(4x-1)^n} \right|
\]

\[
= \lim_{n \to \infty} \frac{|4x-1|^{n+1}}{2^{n+1}\sqrt{n+2}} \cdot \frac{2^n\sqrt{n+1}}{|4x-1|^n}
\]

\[
= \lim_{n \to \infty} \frac{|4x-1|}{2} \sqrt{\frac{n+1}{n+2}}
\]

\[
= \frac{3}{2} |x - \frac{1}{4}|
\]

The series converges when \( \frac{3}{2} |x - \frac{1}{4}| < 1 \), or when \( |x - \frac{1}{4}| < \frac{1}{2} \). This series has radius of convergence \( \frac{1}{2} \) and center \( \frac{1}{4} \), and thus converges when \( -\frac{1}{4} < x < \frac{3}{4} \). We need to check the endpoint of this interval.

When \( x = -\frac{1}{4} \):

\[
\sum_{n=1}^{\infty} \frac{(-1)^n(4x-1)^n}{2^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{2^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{2^n(-1)^n}{2^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}.
\]

This series diverges by the limit comparison test with \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \). (Let \( a_n = \frac{1}{\sqrt{n+1}} \), \( b_n = \frac{1}{\sqrt{n}} \) and show \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 > 0 \). Then since \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges by the p-series test, so does \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \).)

When \( x = \frac{3}{4} \):

\[
\sum_{n=1}^{\infty} \frac{(-1)^n(4x-1)^n}{2^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{2^n(-1)^n}{2^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.
\]

We have \( b_n = \frac{1}{\sqrt{n+1}} > 0 \). Furthermore,

- \( b_{n+1} = \frac{1}{\sqrt{n+2}} \leq \frac{1}{\sqrt{n}} = b_n \) for all \( n \geq 1 \),
\[ \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{\sqrt{n + 1}} = 0. \]

Thus, the series converges by the Alternating Series Test. The interval of convergence is \((-\frac{1}{4}, \frac{3}{4})\).

5. a) Find the 6th Taylor Polynomial of \( f(x) = \sin(x) \) about \( a = \frac{\pi}{2} \).

**Solution:** By definition, the 6th Taylor Polynomial of \( f(x) \) about \( x = a \) is given by

\[
T_6(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5 + \frac{f^{(6)}(a)}{6!}(x-a)^6. 
\]

Here we have \( a = \frac{\pi}{2} \) and we know \( f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \). We compute the first 6 derivatives:

\[
\begin{align*}
  f'(x) &= \cos(x) &\Rightarrow f'\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) = 0 \\
  f''(x) &= -\sin(x) &\Rightarrow f''\left(\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) = -1 \\
  f^{(3)}(x) &= -\cos(x) &\Rightarrow f^{(3)}\left(\frac{\pi}{2}\right) &= -\cos\left(\frac{\pi}{2}\right) = 0 \\
  f^{(4)}(x) &= \sin(x) &\Rightarrow f^{(4)}\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) = 1 \\
  f^{(5)}(x) &= \cos(x) &\Rightarrow f^{(5)}\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) = 0 \\
  f^{(6)}(x) &= -\sin(x) &\Rightarrow f^{(6)}\left(\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) = -1
\end{align*}
\]

So,

\[
T_6(x) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6.
\]

b) Write down the Taylor series expansion of \( \sin(x) \) about \( \frac{\pi}{2} \). Write out at least the first four terms for each series required below in addition to the general formula for the nth
term; for example

\[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \]

**Solution:** Most of the work was done in part a. The key thing to recognize here is that the values 1, 0, −1, 0 repeat. For all odd values of \( n \), we have \( f^{(n)} \left( \frac{\pi}{2} \right) = 0 \). Furthermore, if \( n \) is even we have \( n = 2k \) and

\[
f^{(2k)} \left( \frac{\pi}{2} \right) = \begin{cases} 
1 & \text{if } k \text{ is even (i.e. if } 2k \text{ is divisible by } 4); \\
-1 & \text{if } k \text{ is odd.} 
\end{cases} = (-1)^k
\]

So, our general term is of the form \( \frac{(-1)^k (x - \frac{\pi}{2})^{2k}}{(2k)!} \). Thus,

\[
\sin(x) = 1 - \frac{1}{2} \left( x - \frac{\pi}{2} \right)^2 + \frac{1}{24} \left( x - \frac{\pi}{2} \right)^4 - \frac{1}{720} \left( x - \frac{\pi}{2} \right)^6 + \cdots + \frac{(-1)^k \left( x - \frac{\pi}{2} \right)^{2k}}{(2k)!} + \cdots
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k \left( x - \frac{\pi}{2} \right)^{2k}}{(2k)!}
\]