Worksheet 6, Math 10560

Times indicate the amount of time that you would be expected to spend on the problem in an exam. All problems have appeared on old exams for Calculus 2.

1. (4 mins) Evaluate the integral or show that it is divergent.

\[
\int_{-\infty}^{4} \frac{1}{x^2 - 6x + 10} \, dx
\]

**Solution:** This is an improper integral because the interval of integration is infinite. We complete the square and evaluate the integral as follows:

\[
\int_{-\infty}^{4} \frac{1}{x^2 - 6x + 10} \, dx = \lim_{t \to -\infty} \int_{t}^{4} \frac{1}{(x - 3)^2 + 1} \, dx
\]

\[
= \lim_{t \to -\infty} \left[ \arctan(x - 3) \right]_{t}^{4}
\]

\[
= \lim_{t \to -\infty} \left[ \arctan(1) - \arctan(t - 3) \right]
\]

\[
= \arctan(1) - \lim_{t \to -\infty} \arctan(t - 3)
\]

\[
= \frac{\pi}{4} - \left( -\frac{\pi}{2} \right)
\]

\[
= \frac{3\pi}{4}.
\]
2. (4 mins) Find the arc length of the curve \( y = \ln(\sec x) \) between \( 0 \leq x \leq \frac{\pi}{4} \) using the arc length formula.

**Solution:** First, note that

\[
\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x.
\]

Then by the arc length formula we have

\[
L = \int_{0}^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

\[
= \int_{0}^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} \, dx
\]

\[
= \int_{0}^{\frac{\pi}{4}} \sqrt{\sec^2 x} \, dx
\]

\[
= \int_{0}^{\frac{\pi}{4}} \sec x \, dx
\]

\[
= \ln |\sec x + \tan x| \bigg|_{0}^{\frac{\pi}{4}}
\]

\[
= \ln |\sqrt{2} + 1| - \ln |1 + 0|
\]

\[
= \ln |\sqrt{2} + 1|.
\]
3. (8 mins) Complete the following sentences using the words *converges* and *diverges*:

\[
\int_1^\infty \frac{1}{x^p} \, dx \quad \text{converges} \quad \text{if} \quad p > 1 \quad \text{and} \quad \text{diverges} \quad \text{if} \quad p \leq 1.
\]

\[
\int_0^1 \frac{1}{x^p} \, dx \quad \text{diverges} \quad \text{if} \quad p \geq 1 \quad \text{and} \quad \text{converges} \quad \text{if} \quad p < 1.
\]

Decide whether the following improper integrals converge or diverge by comparing them to a known integral. In each case, state which integral you are comparing the given integral to and state clearly why you can conclude convergence or divergence.

(a) \( \int_1^\infty \frac{1}{x^2 + 3x + 3} \, dx \)

**Solution:** Note that

\[ \frac{1}{x^2 + 3x + 3} \leq \frac{1}{x^2} \]

for all \( x \geq 1 \). Since the integral \( \int_1^\infty \frac{1}{x^2} \, dx \) converges by the p-test, we conclude \( \int_1^\infty \frac{1}{x^2 + 3x + 3} \, dx \) converges as well by the comparison test.

(b) \( \int_1^\infty \frac{1}{x + e^x} \, dx \)

**Solution:** First, we observe that

\[ \frac{1}{x + e^x} < \frac{1}{e^x} \]

for all \( x \geq 1 \). Next, we compute the integral

\[
\int_1^\infty \frac{1}{e^x} \, dx = \lim_{t \to \infty} \int_1^t e^{-x} \, dx
= \lim_{t \to \infty} -e^{-x}\bigg|_1^t
= \lim_{t \to \infty} -e^{-t} + e^{-1}
= \frac{1}{e}.
\]

Thus by the comparison test, since \( \int_1^\infty \frac{1}{e^x} \, dx \) converges, so does \( \int_1^\infty \frac{1}{x + e^x} \, dx \).
4. Let $y' = (x - 2)(y - x)$ with $y(1) = 2$.

(a) Draw a $3 \times 3$ direction field ($x = 1, 2, 3$, $y = 1, 2, 3$), and approximate solution curve.

Solution: We compute $y' = F(x, y) = (x - 2)(y - x)$ at the points $(x, y)$ for $x = 1, 2, 3$, $y = 1, 2, 3$.

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Using this data we can sketch the direction field and an approximate solution to the equation with $y(1) = 2$.

(b) Use Euler’s method with $\Delta x = 1$ to estimate $y(3)$. How close is Euler’s method to the solution curve you drew by hand above?

Solution: We use Euler’s Method with step size $\Delta x = 1$. Our first point is $(x_0, y_0) = (1, 2)$.

$$x_1 = x_0 + \Delta x = 2 \Rightarrow y_1 = y_0 + F(x_0, y_0) = 2 + (1 - 2)(2 - 1) = 1$$
$$x_2 = x_1 + \Delta x = 3 \Rightarrow y_2 = y_1 + F(x_1, y_1) = 1 + (2 - 2)(1 - 2) = 1$$
So $y(3) \approx 1$. The solution drawn in part a gives an estimate of $y(3) \approx 1.2$. 

5. Find the length of the minor arc of the circle \( y^2 + x^2 = 9 \) between the point \((0, 3)\) and \((3, 0)\) using the arc length formula.

**Solution:** First, we solve \( y^2 + x^2 = 9 \) for \( y \) to obtain \( y = \sqrt{9 - x^2} \). Then the derivative with respect to \( x \) is \( \frac{dy}{dx} = \frac{-x}{\sqrt{9 - x^2}} \). Using the arc length formula, we get

\[
L = \int_{0}^{3} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \\
= \int_{0}^{3} \sqrt{1 + \left( \frac{-x}{\sqrt{9 - x^2}} \right)^2} \, dx \\
= \int_{0}^{3} \sqrt{1 + \frac{x^2}{9 - x^2}} \, dx \\
= \int_{0}^{3} \frac{\sqrt{9}}{\sqrt{9 - x^2}} \, dx \\
= 3 \int_{0}^{3} \frac{1}{\sqrt{9 - x^2}} \, dx \\
= 3 \left[ \arcsin \left( \frac{x}{3} \right) \right]_{0}^{3} \\
= 3 \left( \arcsin(1) - \arcsin(0) \right) \\
= 3\left( \frac{\pi}{2} - 0 \right) \\
= \frac{3\pi}{2}.
\]

Verify your answer using the formula from geometry

\[
L = (\text{circumference}) \cdot \left( \frac{\text{angle}}{2\pi} \right) = (2\pi R) \cdot \left( \frac{\theta}{2\pi} \right) = R\theta.
\]

**Solution:** The arc from \((0, 3)\) to \((3, 0)\) corresponds to the angle from 0 to \( \frac{\pi}{2} \). Thus \( \theta = \frac{\pi}{2} \) and

\[
L = 3 \cdot \frac{\pi}{2}
\]
as expected.