Name: ______________________
Section: ______________________

Math 10560, Worksheet 2
January 31, 2023

1. Calcisfunium-127 decays exponentially with a half life of 40 years. Suppose we have a 100-mg sample.

   (a) Find the mass that remains after t years.

   Solution:
   Exponential decay satisfies the formula $m(t) = m_0 e^{kt}$, where $m(t)$ is the mass in mg after t years. The half life means we have the equation:

   $$\frac{1}{2}m_0 = m_0 e^{40k}$$
   $$\frac{1}{2} = e^{40k}$$
   $$\ln\left(\frac{1}{2}\right) = 40k$$

   Giving that our decay constant is $k = \frac{1}{40}\ln\left(\frac{1}{2}\right)$, and so

   $$m(t) = 100e^{\frac{t}{40}\ln\left(\frac{1}{2}\right)} = 100e^{\ln\left(\left(\frac{1}{2}\right)^{\frac{t}{40}}\right)} = 100\left(\frac{1}{2}\right)^{\frac{t}{40}}$$

   (b) How much of the sample remains after 100 years?

   Solution:
   We see that $m(100) = 100\left(\frac{1}{2}\right)^{\frac{100}{40}} = 100\left(\frac{1}{2}\right)^{\frac{5}{2}} \approx 17.67$ mg.

   (c) After how long will only 1 mg remain?

   Solution:
   We need to solve for $m(t) = 1$ mg, giving:

   $$1 = 100e^{\frac{t}{40}\ln\left(\frac{1}{2}\right)}$$
   $$\frac{1}{100} = e^{\frac{t}{40}\ln\left(\frac{1}{2}\right)}$$
   $$\ln\left(\frac{1}{100}\right) = \frac{t}{40}\ln\left(\frac{1}{2}\right)$$
   $$\frac{\ln\left(\frac{1}{100}\right)}{\ln\left(\frac{1}{2}\right)} = \frac{t}{40}$$

   And so $t = 40\frac{\ln\left(\frac{1}{100}\right)}{\ln\left(\frac{1}{2}\right)} = 40\frac{\ln 100}{\ln 2} \approx 265.75$ years.

2. Given $f(x) = (\sqrt{x}-2)^{\log_3 x}$, find the equation of the tangent line to $f(x)$ at $x = 9$.

Solution:
To find the tangent line, we need to know the slope and find a point on the line. At $x = 9$, we have that $f(9) = 1^{\log_3 9} = 1^2 = 1$, giving the point $(9, 1)$ on the line.

To find the slope, we need $f'(9)$, for which we use logarithmic differentiation.

$$\ln f(x) = \ln\left(\left(\sqrt{x}-2\right)^{\log_3 x}\right)$$

$$\ln f(x) = \log_3 x \ln\left(\sqrt{x}-2\right)$$

$$\frac{1}{f(x)} f'(x) = \frac{1}{(\ln 3)x}\ \ln\left(\sqrt{x}-2\right) + \log_3 x\ \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{x}-2}$$

Substituting in $x = 9$, we see:

$$\frac{1}{f(9)} f'(9) = \frac{1}{(\ln 3)9}\ \ln\left(\sqrt{9}-2\right) + \log_3 9\ \frac{\frac{1}{2}9^{-\frac{1}{2}}}{\sqrt{9}-2}$$

$$f'(9) = \frac{1}{(\ln 3)9}\ \ln 1 + 2\ \frac{1}{6} = \frac{1}{3}$$

So we have the equation of the tangent line is:

$$y - 1 = \frac{1}{3}(x-9)$$

3. Evaluate $\displaystyle\int_0^{\frac{1}{2}\ln 5} \frac{e^{2x}}{1+e^{2x}}\ dx$.

Solution:
Performing the substitution $u = 1 + e^{2x},\ du = 2e^{2x}$, we have that:

$$\int_0^{\frac{1}{2}\ln 5} \frac{e^{2x}}{1+e^{2x}}\ dx = \frac{1}{2}\int_2^6 \frac{1}{u}\ du = \frac{1}{2}\ln u\ |_2^6 = \frac{1}{2}(\ln 6 - \ln 2) = \frac{1}{2}\ln 3$$

4. Use implicit differentiation to find $\frac{dy}{dx}$ if $x + \log_2 y = 3^{x+y}$.

Solution:

Taking the derivative of both sides of the equation with respect to x gives:

$$1 + \frac{1}{(\ln 2)\ y} \frac{dy}{dx} = (\ln 3)\ 3^{x+y} \left(1 + \frac{dy}{dx}\right)$$

$$1 + \frac{1}{(\ln 2)\ y} \frac{dy}{dx} = (\ln 3)\ 3^{x+y} + (\ln 3)\ 3^{x+y} \frac{dy}{dx}$$

$$\frac{1}{(\ln 2)\ y} \frac{dy}{dx} - (\ln 3)\ 3^{x+y} \frac{dy}{dx} = (\ln 3)\ 3^{x+y} - 1$$

$$\left(\frac{1}{(\ln 2)\ y} - (\ln 3)\ 3^{x+y}\right) \frac{dy}{dx} = (\ln 3)\ 3^{x+y} - 1$$

$$\frac{dy}{dx} = \frac{(\ln 3)\ 3^{x+y} - 1}{\frac{1}{(\ln 2)\ y} - (\ln 3)\ 3^{x+y}}$$