Name: ______________________________ Section: ____________

# Math 10560, Worksheet 3
## February 7, 2023

**Instruction:** For full credits, please

- do ALL of the questions 1, 2, and 3, and
- do ONE of the questions 4, or 5.

**1.** Evaluate the following integrals:

$$(a) \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\, dx \qquad\qquad (b) \int x^2 \ln x\, dx \quad (x > 0)$$

***Solution:***

(a) Let $u = \arcsin x$, then $du = \dfrac{1}{\sqrt{1-x^2}}\, dx$, and $u(0) = 0$ and $u(1) = \dfrac{\pi}{2}$. Hence, substituting back to the original integral, we have

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\, dx = \int_0^1 \arcsin(x) \cdot \frac{1}{\sqrt{1-x^2}}\, dx = \int_0^{\pi/2} u\, du = \frac{u^2}{2}\Big|_0^{\pi/2} = \frac{\pi^2}{8} - 0 = \frac{\pi^2}{8}.$$

(b) We perform integration by parts. Let

$$\begin{cases} u = \ln x & \Longrightarrow \ du = \frac{1}{x}\, dx \\ dv = x^2 & \Longrightarrow \ v = \frac{x^3}{3} \end{cases}$$

then

$$\begin{aligned} \int x^2 \ln x\, dx &= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x}\, dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2\, dx \\ &= \frac{x^3}{3} \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C, \end{aligned}$$

where $C$ is an arbitrary constant.

**2.** Evaluate the integral $\int e^{2x} \sin x \, dx$.

(*Hint:* perform integration by parts twice, with EITHER $u = e^{2x}$ OR $dv = e^{2x}\,dx$, *but stay with ONE particular way of part-substitutions for both times*, you should be able to reduce to

$$\int e^{2x} \sin x \, dx = f(x) + a \cdot \int e^{2x} \sin x \, dx + C$$

where $f(x)$ does not have any integral inside, and $a \neq 1$ is a constant. You can then "solve" for the integral by rewriting

$$(1-a) \int e^{2x} \sin x \, dx = f(x) + C$$

and dividing both sides by $1 - a$. )

***Solution:*** Following the hint, the first time, we let $u = e^{2x}$ and $dv = \sin x \, dx$, then $du = 2e^{2x}\,dx$ and $v = -\cos x$. Thus,

$$\int e^{2x} \sin x \, dx = -e^{2x} \cos x - \int (-\cos x)\, 2e^{2x}\, dx = -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx.$$

The second time, we let continue to let $u = e^{2x}$ and let $dv = \cos x \, dx$, then $du = 2e^{2x}\,dx$ and $v = \sin x$. Thus,

$$\begin{aligned} \int e^{2x} \sin x \, dx &= -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx \\ &= -e^{2x} \cos x + 2 \left( e^{2x} \sin x - \int \sin x \, (2e^{2x})\, dx \right) \\ &= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx + C. \end{aligned}$$

We see from the hint that the original integral re-appears on the right-hind side (with $f(x) = -e^{2x} \cos x + 2e^{2x} \sin x$ and $a = -4$). Hence, moving the integral of the right-hand side to the left, we have

$$\begin{aligned} 5 \int e^{2x} \sin x \, dx &= -e^{2x} \cos x + 2e^{2x} \sin x + C, \\ \implies \int e^{2x} \sin x \, dx &= \frac{1}{5} \left( -e^{2x} \cos x + 2e^{2x} \sin x \right) + C. \end{aligned}$$

**3.** Use L'Hôpital's rule to compute the following limits. For every time you use the rule, please specify which case it applies to ($\frac{0}{0}$ or $\frac{\infty}{\infty}$).

$$(a)\ \lim_{x\to-\infty} x^3e^{2x} \qquad\qquad (b)\ \lim_{x\to0^+}(1+2x)^{\frac{1}{x}}$$

(*Hint:* For (b), use $1+2x = e^{\ln(1+2x)}$ for $x>0$)

***Solution:***

(a) When $x\to-\infty$, we have $x^3\to-\infty$ and $e^{2x}\to0$, so this is the indeterminate case $0\cdot\infty$. Rewriting the expression $x^3e^{2x}$ as $\dfrac{x^3}{e^{-2x}}$, then we have the indeterminate case $\frac{\infty}{\infty}$. Using L'Hôpital's rule we have

$$\lim_{x\to-\infty}\frac{x^3}{e^{-2x}} = \lim_{x\to-\infty}\frac{(x^3)'}{(e^{-2x})'} = \lim_{x\to-\infty}\frac{3x^2}{-2e^{-2x}}$$

Again, this is the case $\frac{\infty}{\infty}$, so use the rule once more gives

$$\lim_{x\to-\infty}\frac{3x^2}{-2e^{-2x}} = \lim_{x\to-\infty}\frac{(3x^2)'}{(-2e^{-2x})'} = \lim_{x\to-\infty}\frac{6x}{4e^{-2x}},$$

This is again the case $\frac{\infty}{\infty}$, so use the rule another time gives

$$\lim_{x\to-\infty}\frac{6x}{4e^{-2x}} = \lim_{x\to-\infty}\frac{(6x)'}{(4e^{-2x})'} = \lim_{x\to-\infty}\frac{6}{-8e^{-2x}} = \frac{6}{8}\lim_{x\to\infty}e^{-2x} = 6\cdot0 = 0.$$

Since the limit exists, the uses of L'Hôpital's rules above are valid, and we conclude that

$$\lim_{x\to-\infty}x^3e^{2x} = 0.$$

(b) Following the hint, we see that

$$(1+2x)^{\frac{1}{x}} = \left(e^{\ln(1+2x)}\right)^{\frac{1}{x}} = e^{\ln(1+2x)\cdot\frac{1}{x}} = e^{\frac{\ln(1+2x)}{x}}.$$

Since exponential functions are continuous, we have

$$\lim_{x\to0^+}e^{\frac{\ln(1+2x)}{x}} = e^{\lim_{x\to0^+}\frac{\ln(1+2x)}{x}},$$

so it suffices to compute the limit

$$\lim_{x\to0^+}\frac{\ln(1+2x)}{x}.$$

This is an indeterminate case of the form $\frac{0}{0}$, so using L'Hôpital's rule, we have

$$\lim_{x\to0^+}\frac{\ln(1+2x)}{x} = \lim_{x\to0^+}\frac{(\ln(1+2x))'}{(x)'} = \lim_{x\to0^+}\frac{\frac{2}{1+2x}}{1} = \lim_{x\to0+}\frac{2}{1+2x} = \frac{2}{1+0} = 2.$$

Hence, the limit we want to compute is

$$\lim_{x\to0^+}(1+2x)^{1/x} = e^{\lim_{x\to0^+}\frac{\ln(1+2x)}{x}} = e^2.$$

**4.** Consider the limit

$$\lim_{x\to\infty} \frac{x+\sin x}{2x} \qquad (*)$$

(a) Use a graphic calculator or tools such as *desmos.com*, guess the value of this limit $(*)$ (*Hint*: what is a horizontal asymptote of the function $\frac{x+\sin x}{2x}$?)

(b) It is known that $\lim_{x\to\infty}(x+\sin x)=\infty$ (You don't need to verify this). This means the limit $(*)$ has the indeterminate form $\frac{\infty}{\infty}$.
Also, $\lim_{x\to\infty}\sin x$ and $\lim_{x\to\infty}\cos x$ do not exist, as $\sin x, \cos x$ keep fluctuating between $-1$ and $1$.
What happens when you apply L'Hôpital's rule to $(*)$? Does it agree with (a)? What's wrong?

(c) Use the fact that $-1 \le \sin x \le 1$ to show that when $x$ is positive and large

$$\frac{x-1}{2x} \le \frac{x+\sin x}{2x} \le \frac{x+1}{2x}.$$

Now, use Squeeze Theorem to obtain the limit you guessed in part (a).

***Solution:***

(a) From the graph of $f(x)=\dfrac{x+\sin x}{2x}$ we see that $y=\dfrac{1}{2}$ is the horizontal asymptote of $f(x)$ when $x\to\infty$, so we guess that

$$\lim_{x\to\infty}\frac{x+\sin x}{2x}=\frac{1}{2}.$$

(b) Suppose we were to use L'Hôpital's rule, we see that

$$\lim_{x\to\infty}\frac{x+\sin x}{2x} \quad \text{“=”} \quad \lim_{x\to\infty}\frac{1+\cos x}{2} = \lim_{x\to\infty}\left(\frac{1}{2}+\frac{\cos x}{2}\right),$$

which does not exist as $\frac{\cos x}{2}$ will keep fluctuating between $-\frac{1}{2}$ and $\frac{1}{2}$. However, we cannot conclude from this that the original limit does not exist, since the criteria for L'Hôpital's rule does not apply in this case. This also contradicts what we observe from part (a).

We recall that the criteria for using L'Hôpital's rule are that (i) $\frac{f(x)}{g(x)}$ must have indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when $x\to a$, and (ii) the limit

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$$

must exist or $L=\pm\infty$. In this case (ii) is violated.

(c) Following the hint, we have $-1\le\sin x\le 1$, so $x-1\le x+\sin x\le x+1$, so when $x$ is positive, we have

$$\frac{x-1}{2x} \le \frac{x+\sin x}{2x} \le \frac{x+1}{2x}.$$

We can see that

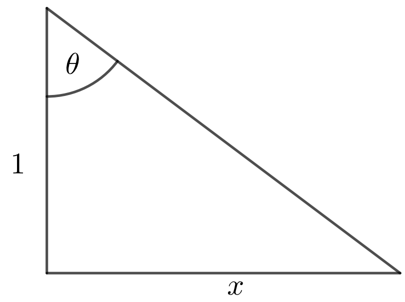
$$\lim_{x\to\infty} \frac{x-1}{2x} = \lim_{x\to\infty} \frac{x+1}{2x} = \lim_{x\to\infty} \frac{1 \pm \frac{1}{x}}{2} = \frac{1}{2}.$$

Hence, by Squeeze Theorem, we also have

$$\lim_{x\to\infty} \frac{x+\sin x}{2x} = \frac{1}{2}.$$

**5.** (a) Use Chain rule to compute the derivative of $\sin(\arctan x)$.

(b) Use the triangle below to simplify the functions $\sin(\arctan x)$ and $\cos(\arctan x)$ so that the final expressions do not contain trigonometry functions of $x$.




(*Hint:* observe that $\tan\theta = x$, so $\arctan x = \theta$, compute the length of the hypotenuse.)

(c) Use the result of part (b) to compute the derivative of $\sin(\arctan x)$ again. How does this agree with the result in part (a)?

***Solution:***

(a) We have

$$(\sin(\arctan x))' = \cos(\arctan x) \cdot (\arctan x)' = \cos(\arctan x) \cdot \frac{1}{1+x^2}.$$

(b) From the figure above, the length of the hypotenuse in the triangle is $\sqrt{1+x^2}$, so

$$\sin(\arctan x) = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}},$$
$$\cos(\arctan x) = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}.$$

(c) From part (b), we have

$$\begin{aligned}(\sin(\arctan x))' &= \left(\frac{x}{\sqrt{1+x^2}}\right)' = \frac{1\cdot\sqrt{1+x^2} - \frac{x\cdot x}{\sqrt{1+x^2}}}{1+x^2} \\ &= \frac{1+x^2-x^2}{(1+x^2)\sqrt{1+x^2}} = \frac{1}{(1+x^2)^{3/2}}.\end{aligned}$$

This agrees with part (a) by noting that from part (b)

$$\cos(\arctan x) \cdot \frac{1}{1+x^2} = \frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}.$$