Name: ____________________

Section: ____________________

Math 10560, Worksheet 4
February 14, 2023

1. (a) Evaluate:

$$\int x\sin(x)dx.$$

Solution: We integrate by parts using $u = x$, $dv = \sin(x)dx$. Then $du = dx$ and $v = -\cos(x)$. Since $\int u\,dv = uv - \int v\,du$, we have

$$\begin{aligned}\int x\sin(x)dx &= -x\cos(x) - \int -\cos(x)dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

(b) Evaluate:

$$\int x\sin(x^2)dx.$$

Solution: We perform the substitution $u = x^2$, $du = 2\,xdx$ to get

$$\begin{aligned}\int x\sin(x^2)dx &= \frac{1}{2}\int \sin(u)du \\ &= -\frac{1}{2}\cos(u) + C \\ &= -\frac{1}{2}\cos(x^2) + C.\end{aligned}$$

(c) Evaluate:

$$\int x \sin^2(x^2) dx.$$

Solution: We perform the same substitution as in part (b) to get

$$\begin{aligned}
\int x \sin^2(x^2) dx &= \frac{1}{2} \int \sin^2(u) du \\
&= \frac{1}{2} \int \frac{1}{2}(1 - \cos(2u)) du \\
&= \frac{1}{4} \left(\int du - \int \cos(2u) du \right) \\
&= \frac{1}{4} \left[u + C_1 - \left(\frac{1}{2} \sin(2u) + C_2 \right) \right] \\
&= \frac{1}{4} x^2 - \frac{1}{8} \sin(2x^2) + C,
\end{aligned}$$

where $C = \frac{1}{4}(C_1 - C_2)$.

2. Evaluate:

$$\int_{\pi/2}^{2\pi} \sin^2(\theta) \cos^3(\theta) d\theta.$$

Solution: Since 3 is odd, we use the substitution $u = \sin(\theta)$ and $du = \cos(\theta) d\theta$. For the bounds, note that if $\theta = \frac{\pi}{2}$, then $u = 1$ and if $\theta = 2\pi$, then $u = 0$. Thus,

$$\begin{aligned}
\int_{\pi/2}^{2\pi} \sin^2(\theta) \cos^3(\theta) d\theta &= \int_{\pi/2}^{2\pi} \sin^2(\theta) \left(1 - \sin^2(\theta)\right) \cos(\theta) d\theta \\
&= \int_1^0 u^2 \left(1 - u^2\right) du \\
&= \int_1^0 \left(u^2 - u^4\right) du \\
&= \left[\frac{1}{3} u^3 - \frac{1}{5} u^5 \right]_1^0 \\
&= -\frac{1}{3} + \frac{1}{5} \\
&= -\frac{2}{15}.
\end{aligned}$$

3. (a) Evaluate:

$$\int_0^5 \frac{1}{\sqrt{x^2+25}}dx.$$

Solution: We use the trig sub $x = 5\tan(\theta)$, $dx = 5\sec^2(\theta)d\theta$, where $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$. For our bounds, note that if $x = 0$, then $\theta = 0$, and if $x = 5$, then $\theta = \frac{\pi}{4}$. Thus,

$$\begin{aligned}
\int_0^5 \frac{1}{\sqrt{x^2+25}}dx &= \int_0^{\pi/4} \frac{5\sec^2(\theta)}{\sqrt{(5\tan(\theta))^2+25}}d\theta \\
&= \int_0^{\pi/4} \frac{5\sec^2(\theta)}{5\sqrt{\tan^2(\theta)+1}}d\theta \\
&= \int_0^{\pi/4} \sec(\theta)d\theta \\
&= \Big[\ln\big|\sec(\theta)+\tan(\theta)\big|\Big]_0^{\pi/4} \\
&= \ln\left|\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right| - \ln\big|\sec(0)+\tan(0)\big| \\
&= \Big(\ln(\sqrt{2}+1) - \ln(1+0)\Big) \\
&= \ln(\sqrt{2}+1).
\end{aligned}$$

(b) Evaluate:

$$\int_1^6 \frac{1}{\sqrt{x^2-2x+26}}dx.$$

Hint: The solution to part (a) might help.

Solution: We complete the square to see that $x^2 - 2x + 26 = (x-1)^2 + 25$, so

$$\int_1^6 \frac{7}{\sqrt{x^2-2x+26}}dx = \int_1^6 \frac{1}{\sqrt{(x-1)^2+25}}dx.$$

Using the substitution $u = x - 1$, our integral becomes

$$\int_0^5 \frac{1}{\sqrt{u^2+25}}du = \ln(\sqrt{2}+1),$$

as we showed in part (a).

4. Use partial fraction decomposition to evaluate:

$$\int \frac{6}{x^2-9}dx.$$

Solution: To find the partial fraction decomposition of the integrand, we write it as

$$\begin{aligned}\frac{6}{x^2-9} &= \frac{6}{(x+3)(x-3)} \\ &= \frac{A}{x+3} + \frac{B}{x-3} \\ &= \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}.\end{aligned}$$

From this, we get $6 = A(x-3) + B(x+3)$. To find A and B, we plug in $x = 3$ and $x = -3$. This first substitution yields $6 = 6B$, so $B = 1$, and the second substitution yields $6 = -6A$, so $A = -1$. Thus,

$$\begin{aligned}\int \frac{6}{x^2-9}dx &= \int \left(\frac{-1}{x+3} + \frac{1}{x-3}\right) dx \\ &= -\int \frac{dx}{x+3} + \int \frac{dx}{x-3} \\ &= -\int \frac{du}{u} + \int \frac{dv}{v} \qquad \text{where } u = x+3, v = x-3 \\ &= -\ln|u| + \ln|v| + C \\ &= \ln\left|\frac{v}{u}\right| + C \\ &= \ln\left|\frac{x-3}{x+3}\right| + C.\end{aligned}$$