Name: ____________________

Math 10560, Worksheet 5
February 28th, 2023

The total is 25 points. Divide by 5 to get a grade out of 5.

1. For each of the following problems, state which technique of integration you would use to solve it. You do not need to compute the integrals.

   9 points (1.5 points for each integral)

   (a) $\int x^3 \ln x \, dx$

   Solution: Integration by parts, with $u = \ln x$ and $v' = x^3$.

   (b) $\int \frac{7w}{(w-5)^2} \, dw$

   Solution: Integration by substitution, with $u = w - 5$.

   (c) $\int \frac{\tan^3 x}{\cos^3 x} \, dx$

   Solution: Integration by substitution, with $u = \cos x$.

   (d) $\int e^x \sqrt{1 + e^x} \, dx$

   Solution: Integration by substitution, with $u = 1 + e^x$.

   (e) $\int \sqrt{3 - 2x - x^2} \, dx$

   Solution: Complete the square to get $\sqrt{4 - (x+1)^2}$ and then use trigonometric substitution.

   (f) $\int \ln(x^2 - 1) \, dx$

Solution: Integration by parts, with $u = \ln(x^2 - 1)$ and $v' = 1$.

2. (5 points) Compute the following definite integral:

$$\int_0^1 \frac{1}{(1+x^2)^{3/2}}\, dx$$

Solution:

We use the substitution $x = \tan\theta$. Then $dx = \sec^2\theta\ d\theta$.

Note that $0 = \tan\theta$ and $1 = \tan(\pi/4)$ and so the substitution gives us:

$$\int_0^1 \frac{1}{(1+x^2)^{3/2}}\, dx = \int_0^{\pi/4} \frac{1}{(1+\tan^2\theta)^{3/2}} \sec^2\theta\ d\theta = \int_0^{\pi/4} \frac{1}{\sec^3\theta} \sec^2\theta\ d\theta = \int_0^{\pi/4} \frac{1}{\sec\theta}\ d\theta =$$

$$= \int_0^{\pi/4} \cos\theta\ d\theta = [\sin\theta]_0^{\pi/4} = \sin(\pi/4) - \sin 0 = \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

(Note that we used the formula $1 + \tan^2\theta = \sec^2\theta$.)

2 point for substitution; 1 point for correct limits; 2 points for integration

3. (6 points) Compute the integral

$$\int \frac{x^2+3x+8}{(x-3)(x^2+4x+5)}dx \ .$$

Solution: Since $x-3$ is linear and x^2+4x+5 is irreducible, the partial fraction decomposition of the integrand has form

$$\frac{x^2+3x+8}{(x-3)(x^2+4x+5)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4x+5} \qquad (1).(2 points)$$

Multiplying both sides of equation (1) by $(x-3)(x^2+4x+5)$ yields

$$x^2+3x+8 = (x^2+4x+5)(A) + (x-3)(Bx+C) \qquad (2).$$

If we let $x = 3$ in equation (2), we can immediately solve for A:

$$\begin{aligned} 3^2+3\cdot 3+8 &= (3^2+4\cdot 3+5)A + 0(Bx+C) \\ 26 &= 26\cdot A \\ A &= 1. \end{aligned}$$

Putting $A = 1$ into equation (2) and grouping like terms together, we have

$$\begin{aligned} x^2+3x+8 &= (x^2+4x+5)(1) + (x-3)(Bx+C) \\ x^2+3x+8 &= (x^2+4x+5) + (Bx^2+Cx-3Bx-3C) \\ 0x^2+(-1)\cdot x+3 &= Bx^2+(C-3B)x-3C \qquad (3). \end{aligned}$$

From (3) we can see that $B = 0$, and $3 = -3\cdot C \implies C = -1$.(1 point each for A, B, and C) Now we are ready to solve the integral:

$$\begin{aligned} \int \frac{x^2+3x+8}{(x-3)(x^2+4x+5)}dx &= \int \left(\frac{1}{x-3} + \frac{-1}{x^2+4x+5}\right)dx \\ &= \ln|x-3| + \int \frac{-dx}{x^2+4x+5} \\ &= \ln|x-3| - \int \frac{dx}{(x+2)^2+1} \\ &= \ln|x-3| - \tan^{-1}(x+2) + C.(1 point) \end{aligned}$$

4. (3 points) Compute the following integral:

$$\int \frac{1}{x\sqrt{4-x^2}}dx\ .$$

Solution: The presence of the term $\sqrt{4-x^2}$ clues us in to the use of a trig substitution. We label a right triangle with angle θ, so that the opposite edge has length x, the adjacent edge has length $\sqrt{4-x^2}$, and hypotenuse has length 2. Using trig, $x = 2\sin\theta$, so that $= 2\cos\theta d\theta$, so that $dx = 2\cos\theta$, and also $\sqrt{4-x^2} = 2\cos\theta$. Substituting,

$$\int \frac{1}{x\sqrt{4-x^2}}dx \ = \int \frac{2\cos\theta d\theta}{(2\sin\theta)(2\cos\theta)}dx = \frac{1}{2}\int \csc\theta d\theta$$

(2 points for a valid substitution)

Consulting our formula sheet, we find

Finally, we must convert this back into an expression of x. Consulting the triangle once more, we see that $\csc\theta = \frac{2}{x}$ and $\cot\theta = x\sqrt{4-x^2}$. Altogether, then

$\frac{1}{2}\ln|\csc\theta - \cot\theta| + C = \frac{1}{2}\ln|\frac{2-\sqrt{4-x^2}}{x}| + C = \ln|2-\sqrt{4-x^2}| - \ln|x| + C$
(1 point for solving)

5. Compute the value of $\arcsin(\sin(\frac{5\pi}{4}))$ (2 points)

Solution:

$\sin(\frac{5\pi}{4}) = -\sqrt{\frac{1}{2}}$ (1 point)

$\arcsin\big(-\sqrt{\frac{1}{2}}\big) = -\frac{\pi}{4}$(1 point)