Name: ______________________________ Section: __________

Math 10560, Worksheet 6
March 7, 2023

1. Evaluate the integral or show that it is divergent

$$\int_{-\infty}^{2} (2x+1)e^x dx$$

Solution: This is an improper integral of type 1(b), so we have that

$$\int_{-\infty}^{2} (2x+1)e^x dx = \lim_{t\to-\infty} \int_{t}^{2} (2x+1)e^x dx$$

We use integration by parts with $u = 2x+1$ and $dv = e^x dx$, giving $du = 2dx$ and $v = e^x$:

$$\begin{aligned}\int_{t}^{2} (2x+1)e^x dx &= (2x+1)e^x|_t^2 - \int_t^2 2e^x dx \\ &= 5e^2 - (2t+1)e^t - 2(e^2 - e^t) \\ &= 3e^2 - 2te^t + e^t\end{aligned}$$

We know that $e^t \to 0$ as $t \to -\infty$ and by L'Hopital's Rule we have

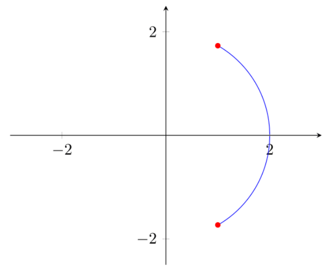
$$\lim_{t\to-\infty} 2te^t = 2\lim_{t\to-\infty} \frac{t}{e^{-t}} = 2\lim_{t\to-\infty} \frac{1}{-e^{-t}} = 2\lim_{t\to-\infty} -e^t = 0$$

Therefore

$$\begin{aligned}\int_{-\infty}^{2} (2x+1)e^x dx &= \lim_{t\to-\infty} \left(3e^2 - 2te^t + e^t\right) \\ &= 3e^2 - 0 + 0 = \boxed{3e^2}\end{aligned}$$

2. Find the arc length of the minor arc of the curve $y^2 + x^2 = 4$ between the points $(1, -\sqrt{3})$ and $(1, \sqrt{3})$, using the arc length formula.

Solution: The portion of the circle we are interested in is given in the following graph:



We can solve the equation for x to get the function $x = \sqrt{4 - y^2}$ (taking the positive square root because x is positive on the portion we are interested in).

Then by the arc length formula we have

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = 2\int_0^{\sqrt{3}} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

where the last equality comes from symmetry of the problem.

Taking the derivative we obtain

$$\frac{dx}{dy} = \frac{-2y}{2\sqrt{4 - y^2}} = \frac{-y}{\sqrt{4 - y^2}}$$

Thus,

$$\begin{aligned}
L &= 2\int_0^{\sqrt{3}} \sqrt{1 + \left(\frac{-y}{\sqrt{4 - y^2}}\right)^2}\, dy \\
&= 2\int_0^{\sqrt{3}} \sqrt{1 + \frac{y^2}{4 - y^2}}\, dy \\
&= 2\int_0^{\sqrt{3}} \sqrt{\frac{4}{4 - y^2}}\, dy \\
&= 4\int_0^{\sqrt{3}} \frac{1}{\sqrt{4 - y^2}}\, dy \\
&= 4 \arcsin\left(\frac{y}{2}\right)\Big|_0^{\sqrt{3}} \\
&= 4\left(\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(0)\right) \\
&= 4 \arcsin\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{4\pi}{3}}
\end{aligned}$$

3. Complete the following sentences using the words *converges* and *diverges* :

$\int_1^\infty \frac{1}{x^p}dx$ ____*converges*____ if $p > 1$ and ____*diverges*____ if $p \leq 1$.

$\int_0^1 \frac{1}{x^p}dx$ ____*diverges*____ if $p \geq 1$ and ____*converges*____ if $p < 1$.

Decide whether the following improper integrals converge or diverge by comparing them to a known integral. In each case, state which integral you are comparing the given integral to and state clearly why you can conclude convergence or divergence.

(a) $\int_1^\infty \frac{1}{x^2+2x+3}\, dx$

> ***Solution:*** Note that $\frac{1}{x^2+3x+3} \leq \frac{1}{x^2}$ for all $x \geq 1$. Using the first statement above with $p = 2$ we have that $\int_1^\infty \frac{1}{x^2}dx$ converges, so by the comparison test $\int_1^\infty \frac{1}{x^2+2x+3}dx$ also converges.

(b) $\int_1^\infty \frac{1}{x-e^{-x}}dx$

> ***Solution:*** Note that $\frac{1}{x} \leq \frac{1}{x-e^{-x}}$ for all $x \geq 1$. Using the first statement above with $p = 1$ we have that $\int_1^\infty \frac{1}{x}dx$ diverges, so by the comparison test $\int_1^\infty \frac{1}{x-e^{-x}}dx$ also diverges.

(c) $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$

> ***Solution:*** We first decompose the integral into two parts:
>
> $$\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx = \int_0^1 \frac{1}{x^2+\sqrt{x}}dx + \int_1^\infty \frac{1}{x^2+\sqrt{x}}dx$$
>
> For non-negative numbers, we always have $x^2+\sqrt{x}$ is greater than both x^2 and $\sqrt{x}$, so $\frac{1}{x^2+\sqrt{x}} \leq \frac{1}{x^2}$ and $\leq \frac{1}{\sqrt{x}}$.
>
> By the second statement above with $p = \frac{1}{2}$ and the first with $p = 2$ we have $\int_0^1 \frac{1}{\sqrt{x}}dx$ and $\int_1^\infty \frac{1}{x^2}dx$ both converge. Then by the comparison test, both pieces of the decomposition converge, so $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$ converges.

4. Let $y' = (x-y)(2-x)$ with $y(1) = 2$.

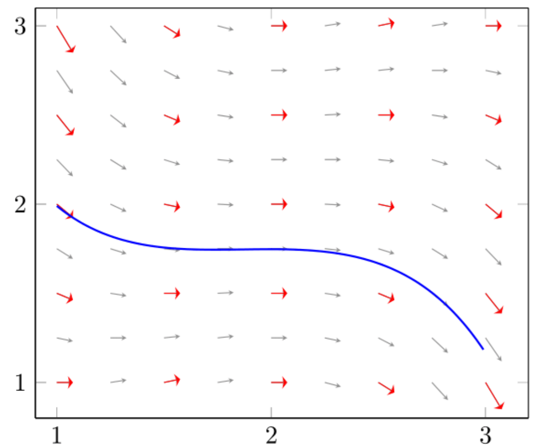
(a) Draw a 5×5 direction field ($x = 1, 1.5, 2, 2.5, 3$, $y = 1, 1.5, 2, 2.5, 3$), and approximate solution curve.

Solution: We compute $y' = F(x,y) = (x-y)(2-x)$ at the points (x,y) for $x = 1, 2, 3$, $y = 1, 2, 3$.

x	1	1	1	1	1	1.5	1.5	1.5	1.5	1.5	2	2	2	2	2
y	1	1.5	2	2.5	3	1	1.5	2	2.5	3	1	1.5	2	2.5	3
y'	0	-0.5	-1	-1.5	-2	0.25	0	-0.25	-0.5	-0.75	0	0	0	0	0

x	2.5	2.5	2.5	2.5	2.5	3	3	3	3	3
y	1	1.5	2	2.5	3	1	1.5	2	2.5	3
y'	-0.75	-0.5	-0.25	0	0.25	-2	-1.5	-1	-0.5	0

Using this data we can sketch the direction field and an approximate solution to the equation with $y(1) = 2$.



(b) Use Euler's method with $\Delta x = 0.5$ to estimate $y(3)$. How close is Euler's method to the solution curve you drew by hand above?

Solution:

We use Euler's Method with step size $\Delta x = 0.5$. Our first point is $(x_0, y_0) = (1, 2)$.

$$
\begin{aligned}
x_1 = x_0 + \Delta x = 1.5 &\Rightarrow y_1 = y_0 + \Delta x * F(x_0, y_0) = 2 + 0.5(-1) = 1.5 \\
x_2 = x_1 + \Delta x = 2 &\Rightarrow y_2 = y_1 + \Delta x * F(x_1, y_1) = 1.5 + 0.5(0) = 1.5 \\
x_3 = x_2 + \Delta x = 2.5 &\Rightarrow y_3 = y_2 + \Delta x * F(x_2, y_2) = 1.5 + 0.5(0) = 1.5 \\
x_4 = x_3 + \Delta x = 3 &\Rightarrow y_4 = y_3 + \Delta x * F(x_3, y_3) = 1.5 + 0.5(-0.5) = 1.25
\end{aligned}
$$

So $y(3) \approx 1.25$. The solution drawn in part a gives an estimate of $y(3) \approx 1.2$.