Name: ______________________________ Section: ____________

**Math 10560, Worksheet 8**
**March 28th, 2023**

Total: 20 pts.

**1.** Determine whether the following series are convergent or divergent, and explain why.

(a) (2 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+2n^2+10}{n^5+n^3+\ln n}$

> ***Solution:*** We have $\frac{n^3+2n^2+10}{n^5+n^3+\ln n} \le \frac{n^3+2n^2+10}{n^5}$. Hence,
>
> $$\frac{n^3+2n^2+10}{n^5+n^3+\ln n} \le \frac{n^3+2n^2+10}{n^5} = \frac{n^3}{n^5} + \frac{2n^2}{n^5} + \frac{10}{n^5} = \frac{1}{n^2} + 2\frac{1}{n^3} + 10\frac{1}{n^5}.$$
>
> Recall the $p$-series test: $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$ for all $p > 1$.
> Therefore, it follows from the comparison test.
> Rubric: 1 pt for the inequality $\frac{n^3+2n^2+10}{n^5+n^3+\ln n} \le \frac{n^3+2n^2+10}{n^5}$ and 1 pt for the conclusion.

(b) (2 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^e} + \frac{2\cdot 4^n}{11^n}$

> ***Solution:*** First, $\sum_{n=1}^{\infty} \frac{1}{n^e} + \frac{2\cdot 4^n}{11^n} = \sum_{n=1}^{\infty} \frac{1}{n^e} + 2\sum_{n=1}^{\infty} \frac{\cdot 4^n}{11^n}$.
> Notice that $e > 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^e} < \infty$. On the other hand, $\sum_{n=1}^{\infty} \frac{4^n}{11^n} = \sum_{n=1}^{\infty} \left(\frac{4}{11}\right)^n$ is a geometric series, so it converges. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^e} + \frac{2\cdot 4^n}{11^n} < \infty$.
> Rubric: 1 pt for $\sum_{n=1}^{\infty} \frac{1}{n^e} < \infty$ and 1 pt for $\sum_{n=1}^{\infty} \frac{2\cdot 4^n}{11^n} < \infty$.

(c) (2 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^{1/2} - n^{1/4}}$

> ***Solution:*** Since $\lim_{n\to\infty} \frac{n^{1/2}}{n^{1/2}-n^{1/4}} = 1$, the series diverges.
> Rubirc: 1 pt for computing the limit $\lim_{n\to\infty} \frac{n^{1/2}}{n^{1/2}-n^{1/4}} = 1$ and 1 pt for the conclusion.

(d) (2 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(2023\cdot n)}{n^2}$

> ***Solution:*** Since $\sin^2 x \le 1$, $\frac{\sin^2(2023\cdot n)}{n^2} \le \frac{1}{n^2}$, and it follows from comparison test.
> Rubirc: 1 pt for the inequality $\frac{\sin^2(2023\cdot n)}{n^2} \le \frac{1}{n^2}$ and 1 pt for the conclusion.

**2.** (4 pts) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges. (*Hint:* Show that $\ln n < n$ for $n$ large.)

> ***Solution:*** First, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$, so for $n$ very large, $\frac{\ln n}{n}$ is a very small number, in particular, $\ln n < n$ for $n$ large. Another way to see this is to consider the function $f(x) = \ln x - x$, $x \in [1, \infty)$. $f'(x) = 1/x - 1 \leq 0$ for $x \geq 1$, and $f(1) = -1 < 0$, so $f$ is negative on $[1, \infty)$, that is, $\ln x < x$ for all $x \geq 1$.
> Now, $\frac{\ln n}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}$, and it follows from comparison test.
> Rubric: 3 pts for showing that $\ln n \leq n$, and 1 pt for the comparison test.

**3.** (4 pts) Consider the following sequences:

$$(I) \quad \left\{(-1)^n \frac{n^3+1}{5n^3+n^2+2}\right\}_{n=1}^{\infty} \qquad (II) \ \left\{\frac{\ln n}{2^n}\right\}_{n=1}^{\infty} \qquad (III) \ \left\{\sin(1/n)\right\}_{n=1}^{\infty}$$

Determine which ones converge and find the limit if the sequence converges.

> ***Solution:*** $(I)$ diverges for $\lim_{n\to\infty} \frac{n^3+1}{5n^3+n^2+2} = \frac{1}{5}$. For $(II)$, we use L'Hopital's rule:
>
> $$\lim_{n\to\infty} \frac{\ln n}{2^n} = \lim_{n\to\infty} \frac{1/n}{2^n \ln 2} = \lim_{n\to\infty} \frac{1}{\ln 2 \cdot n \cdot 2^n} = 0.$$
>
> Finally, $(III)$ converges to 0 since sin is a continuous function.
> Rubric: 1 pt for computing $\lim_{n\to\infty} \frac{n^3+1}{5n^3+n^2+2} = \frac{1}{5}$, 1 pt for the conclusion of $(I)$, 1 pt for $(II)$ and 1 pt for $(III)$.

**4.** (4 pts) Consider the following improper integrals:

$$(I) \quad \int_1^{\infty} \frac{1}{x^5}\, dx \qquad (II) \ \int_0^1 \frac{1}{x}\, dx \qquad (III) \ \int_0^{\infty} \frac{1}{x^2}\, dx$$

Determine which ones converge and find the value if the integral converges.

> ***Solution:*** $(I)$ converges and the integral is equal to $1/4$. Both of $(II)$ and $(III)$ diverges. For $(III)$, we write
>
> $$\int_0^{\infty} \frac{1}{x^2}\, dx = \int_0^1 \frac{1}{x^2}\, dx + \int_1^{\infty} \frac{1}{x^2}\, dx,$$
>
> and $\displaystyle\int_0^1 \frac{1}{x^2}\, dx$ diverges.
> Rubric: 1 pt for $(I)$, 1 pt for (II). For $(III)$, 1 pt for beaking the integral into 2 parts and 1 pt for the conclusion.