Name: ____________________
Section: ____________________

Math 10560, Worksheet 9
April 4, 2023

1. (a) Complete the following statements with ">, ≥, <, and ≤".

   1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for p ____ 1 and diverges for p ____ 1.

   2. $\sum_{n=1}^{\infty} ar^{n-1}$ converges for $|r|$ ____ 1 and diverges for $|r|$ ____ 1.

Solution:

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p>1$ and diverges for $p \leq 1$.

2. $\sum_{n=1}^{\infty} ar^{n-1}$ converges for $|r|<1$ and diverges for $|r| \geq 1$.

(b) Finish the statement of the Comparison Test.

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If b_n is convergent and a_n ____ b_n for all n, then $\sum a_n$ is ____________.

2. If b_n is divergent and a_n ____ b_n for all n, then $\sum a_n$ is ____________.

Solution:

1. If b_n is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent.

2. If b_n is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

(c) Use the Comparison Test to determine the convergence or divergence of the following series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\left|\cos(n^{12345})\right|}{n^8+7}$

Solution: Cosine is always between -1 and 1, so $\left|\cos(n^{12345})\right| \le 1$, meaning that $\dfrac{\left|\cos(n^{12345})\right|}{n^8+7} \le \dfrac{1}{n^8+7} < \dfrac{1}{n^8}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^8}$ is a p-series with $p > 1$, it converges, and $\dfrac{\left|\cos(n^{12345})\right|}{n^8+7}$ converges by the Comparison Test.

(ii) $0.1 + \dfrac{0.2}{3} + \dfrac{0.3}{3^2} + \dfrac{0.4}{3^3} + \cdots + \dfrac{0.9}{3^8} + \dfrac{0.10}{3^9} + \dfrac{0.11}{3^{10}} + \cdots + \dfrac{0.99}{3^{98}} + \dfrac{0.100}{3^{99}} + \cdots$

Solution:

For all n, the nth term of this series is less than the nth term of $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$, which is a convergent geometric series. Thus, this series converges by the Comparison Test.

(iii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2}+1\right)^{n-1}$

Hint: What do you know about $\sum 1^{n-1}$?

Solution: For all n, $\left(\dfrac{1}{n^2}+1\right)^{n-1} > 1^{n-1}$. Since $\displaystyle\sum_{n=1}^{\infty} 1^{n-1}$ is a divergent geometric series, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2}+1\right)^{n-1}$ diverges by the Comparison Test.

2. (a) Complete the statement of the Alternating Series Test.

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, with $b_n > 0$, satisfies

(i) ______________________________

(ii) ______________________________

then the series converges.

Solution: The two conditions are:

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

(b) Use the Alternating Series Test or the Divergence Test to determine if the following series converge or diverge.

(i) $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n^2+5n}{7n^5}\right)$

Solution:

To check condition *(i)*, note that $\frac{n^2+5n}{7n^5} = \frac{1+5/n}{7n^3}$. In this form, as n increases, the numerator monotonically decreases and the denominator monotonically increases. Thus, $\frac{(n+1)^2+5(n+1)}{7(n+1)^5} \leq \frac{n^2+5n}{7n^5}$ for all n.

Condition (ii) is also met since $\frac{n^2+5n}{7n^5} = \frac{1+5/n}{7n^3}$ approaches 0 for large n.

Thus, this series converges by AST.

(ii) $\sum_{n=1}^{\infty}(-1)^{n-1}\left(e^{1/n}\right)$

Solution:

Since $\lim_{n\to\infty} e^{1/n} = 1$, $\lim_{n\to\infty}(-1)^{n-1}\left(e^{1/n}\right)$ does not exist. Thus, this series diverges by the Divergence Test.

(*iii*) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\sin\left(\frac{\pi}{2n}\right)$

Solution: To test condition (i), let $f(x)=\sin\left(\frac{\pi}{2x}\right)$. Then, $f'(x)=\frac{-\pi}{2x^2}\cos\left(\frac{\pi}{2x}\right)$. When $x\geq 1$, $f'(x)\leq 0$. Thus, $\sin\left(\frac{\pi}{2n}\right)$ is monotonically decreasing as n increases, and condition (i) is met.
Condition (ii) is also met because $\lim_{n\to\infty}\sin\left(\frac{\pi}{2n}\right)=\sin(0)=0$.
Thus, this series converges by AST.

3. The following series converge (you do not need to show this). For each one, determine if it converges absolutely or conditionally and explain your reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$

Solution: This converges conditionally, since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges.

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{4}{7}\right)^n$

Solution: This converges absolutely since $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{7}\right)^n$ is a convergent geometric series.

(c) $\displaystyle\sum_{n=2}^{\infty}\frac{10}{4n^2-4}$

Solution: We know immediately that the convergence is absolute since all of the terms in the series are positive.

4. (a) Complete the statement of the Ratio Test.

Let $\sum_{n=1}^{\infty} a_n$ be a series with $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

If L ____ 1, then the series converges absolutely.

If L ____ 1, then the series diverges.

If L ____ 1, then the ratio test is inconclusive.

Solution:
If $L<1$, then the series converges absolutely.

If $L>1$, then the series diverges.

If $L=1$, then the ratio test is inconclusive.

(b) What can you determine about the following series from applying the ratio test?

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$

Solution: In this case, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} = 1$ by l'Hopital's Rule. Thus, the Ratio Test is inconclusive.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{n!}$

Solution: Here, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^5 \cdot n!}{(n+1)! \cdot n^5} = \lim_{n\to\infty} \frac{(n+1)^5}{(n+1)\cdot n^5} = \lim_{n\to\infty} \frac{(n+1)^4}{n^5} = 0$ by l'Hopital's Rule. Since $0 < 1$, the series converges by the Ratio Test.

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n^4}$

Solution: Here, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4^{(n+1)} \cdot n^4}{(n+1)^4 \cdot 4^n} = \lim_{n\to\infty} \frac{4n^4}{(n+1)^4} = 4 \lim_{n\to\infty} \frac{n^4}{(n+1)^4} = 4$ by l'Hopital's Rule. Since $4 > 1$, the series diverges by the Ratio Test.

5. Consider the alternating series $\frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{3^2} + \frac{1}{4} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{5^2} + \cdots$.

(a) Does the Alternating Series Test apply here? Why or why not?

Solution: It does not, since the absolute values of the terms are not monotonically decreasing.

(b) Does the Divergence Test tell us anything about the behavior of this series?

Solution: No, it is inconclusive since the limit of the absolute values of the terms is 0.

(c) Notice that we can express this series as $\sum_{n=2}^{\infty} \frac{1}{n} - \frac{1}{n^2}$. What can we learn from the Integral Test?

Solution: By the integral test, $\sum_{n=2}^{\infty} \frac{1}{n} - \frac{1}{n^2}$ diverges if $\int_2^{\infty} \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ diverges. We can break this into two integrals: $\int_2^{\infty} \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \int_2^{\infty} \frac{1}{x} dx - \int_2^{\infty} \frac{1}{x^2} dx$. The first term diverges and the second term converges, so their difference diverges. Thus, $\sum_{n=2}^{\infty} \frac{1}{n} - \frac{1}{n^2}$ diverges.