Name: ______________________________

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1. (13 pts.) A tank has pure water flowing into it at 10 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 L/min. Initially, the tank contains 10 kg of salt in 100 L of water.
How much salt will there be in the tank after 30 minutes?

Mixing Tank Separable Differential Equations Examples

When studying separable differential equations, one classic class of examples is the mixing tank problems. Here we will consider a few variations on this classic.

Example 1. A tank has pure water flowing into it at 10 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 l of water.

How much salt will there be in the tank after 30 minutes?

To study such a question, we consider *the rate of change of the amount of salt in the tank*. Let S be the amount of salt in the tank at any time t. If we can create an equation relating $\frac{dS}{dt}$ to S and t, then we will have a differential equation which we can, ideally, solve to determine the relationship between S and t.

To describe $\frac{dS}{dt}$, we use the concept of *concentration*, the amount of salt per unit of volume of liquid in the tank. In this example, the inflow and outflow rates are the same, so the volume of liquid in the tank stays constant at 100 l. Hence, we can describe the concentration of salt in the tank by

$$\text{concentration of salt} = \frac{S}{100}\,\text{kg/l}.$$

Then, since mixture leaves the tank at the rate of 10 l/min, salt is leaving the tank at the rate of

$$\frac{S}{100}(10\,\text{l/min}) = \frac{S}{10}.$$

This is the rate at which salt *leaves* the tank, so

$$\frac{dS}{dt} = -\frac{S}{10}.$$

This is the differential equation we can solve for S as a function of t. Notice that since the derivative is expressed in terms of a single variable, it is the simplest form of separable differential equations, and can be solved as follows:

$$\int \frac{dS}{S} = -\int \frac{1}{10}\,dt$$

$$\ln|S| = -\frac{1}{10}t + C$$

$$S = Ce^{-\frac{1}{10}t}$$

where C is a positive constant. Note that we have used the fact that $S \geq 0$ to eliminate the absolute value symbol.

Since $S = 10$ when $t = 0$, we find $C = 10$ and finally we have

$$S = 10e^{-\frac{1}{10}t}.$$

We can see from this that as t goes to infinity, the amount of salt in the tank goes to zero. Also, after 30 minutes, there will be

$$S = 10e^{-3} = 0.49787068\,\text{kg}$$

of salt in the tank.