Lecture 2: Section 1.2: Exponents and Radicals

Positive Integer Exponents

If a is any real number and n is any natural number (positive integer), the nth power of a is defined as

$$a^n = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{\text{n factors}}$$

Example Evaluate the following:

$$(-2)^3, \quad 2^4, \quad (-1)^5 \quad (-1)^4, \quad \left(\frac{1}{3}\right)^3, \quad 0^3$$

the number a is called the **base** and n is called the **exponent**.

Integer Exponents

To make the above definition work for exponents which are 0 or negative integers, we must restrict the possibilities for the base to non-zero real numbers.

Zero and Negative Exponents

If $a \neq 0$ is any non-zero real number and n is a positive integer, then we define

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

Example Evaluate the following

$$(-2)^0, \quad 2^{-3}, \quad (-3)^{-2} \quad \left(\frac{1}{3}\right)^{-2}.$$

Laws of Exponents for Integer Exponents The following algebraic rules apply to exponents:

Rule	Example	Description
1. $a^m a^n = a^{m+n}$	$3^5 3^{10} = 3^{15}$	When multiplying two powers of the same number, add the powers.
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^{10}} = 3^{5-10} = 3^{-5}$	When dividing two powers of the same number, subtract the exponents.
3. $(a^m)^n = a^{mn}$	$(3^5)^{10} = 3^{50}$	When raising a power of a number to a new power, multiply the powers.
4. $(ab)^n = a^n b^n$	$(3 \cdot 5)^{10} = 3^{10} \cdot 5^{10}$	A product of two numbers raised to a given power, is the same as the product of the factors raised to the given power.
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{5}\right)^{10} = \frac{3^{10}}{5^{10}}$	Raising a quotient of two numbers to a given power is the same as raising the numerator and the denominator to the given power.
6. $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$	$\left(\frac{3}{5}\right)^{-10} = \frac{5^{10}}{3^{10}}$	To raise a fraction to a negative power, you can flip the fraction and change the sign of the exponent.

The proofs of these rules are not very difficult and can be found in the textbook.
For example to prove the first law when m and n are positive integers, we see that

$$a^m a^n = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{\text{n factors}}\underbrace{a \cdot a \cdot a \cdot a \cdot \dots \cdot a}_{\text{m factors}} = \underbrace{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot \dots \cdot a}_{m+n \text{ factors}} = a^{m+n}$$

Example Simplify the following expressions (here x can have any real number value and z and y can have only non-zero values):

(a) $x^3x^2 + x^4x^1$.

(b) $\dfrac{(x^3)^2x^4}{x^5}$

(c) $\left(\dfrac{xy}{z}\right)^3\dfrac{x^2z^2}{zy}$.

(d) $\left(\dfrac{y}{z}\right)^{-3}\dfrac{x^2z^2}{zy}$.

Scientific Notation A positive number x is said to be written in scientific notation if it is expressed as

$$x = a \times 10^n$$

where $1 \leq a < 10$ and n is an integer.

Example Write the following numbers in scientific notation:

$$.000004, \qquad 100.12$$

Example Write the following numbers in decimal form:

$$2.31 \times 10^{-3}, \qquad 1.31 \times 10^3$$

Radical Powers

Square Roots For any real number $a \geq 0$, we define $a^{1/2}$ in the following way

$$a^{1/2} = b \qquad \text{means} \qquad b^2 = a \quad \text{and} \quad b \geq 0.$$

We also use the notation $\sqrt{a}$ to denote $a^{1/2}$ and we call $\sqrt{a}$, the square root of a.

Example
$\sqrt{9} = 3$ because $3^2 = 9$ and $3 > 0$.

$\sqrt{16} = 4$ because $4^2 = 16$ and $4 > 0$.

For a real number a, and a positive integer n, we define $a^{1/n}$ in a similar way

$$a^{1/n} = b \qquad \text{means} \qquad b^n = a.$$

If n is even, we must have $a \geq 0$ and $b \geq 0$ in this definition.
We also use the notation $\sqrt[n]{a} = a^{1/n}$ and refer to $\sqrt[n]{a}$ as the nth root of a.

Example Evaluate the following:

$$\sqrt[3]{-8}, \qquad \sqrt[4]{16} \qquad \sqrt[5]{-32}$$

Properties of nth Roots The algebraic rules of exponents given above for integer exponents work for n th roots. Since we have not defined a^x where x is a general rational number yet, we first generalize rules 4, 5 and 3 above.

Rule	**Example**
(similar to rule 4 above) 1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ or $(ab)^{1/n} = a^{1/n}b^{1/n}$	$(27 \times 8)^{1/3} = (27)^{1/3}(8)^{1/3} = 3 \times 2 = 6$
(similar to rule 5 above) 2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ or $\left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}}$	$\left(\frac{27}{8}\right)^{1/3} = \frac{(27)^{1/3}}{(8)^{1/3}} = \frac{3}{2}$
(similar to rule 3 above) 3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ or $(a^{1/n})^{1/m} = a^{1/(mn)}$	$\sqrt[3]{\sqrt[2]{64}} = \sqrt[3]{8} = 2 = \sqrt[6]{64}$
4. $\sqrt[n]{a^n} = a$ if n is odd.	$\sqrt[3]{(-3)^3} = -3$
5. $\sqrt[n]{a^n} = \lvert a \rvert$ if n is even.	$\sqrt[2]{(-3)^2} = 3$

Example Simplify the following:

$$\sqrt[3]{27x^3y^6}, \qquad \sqrt{500} + \sqrt{18}$$

Rational Exponents

For any rational exponent (fractional exponent) m/n, where m and n are integers, $n > 0$ and the fraction is in lowest terms, we define

$$a^{m/n} = (\sqrt[n]{a})^m \qquad \text{or equivalently} \qquad a^{m/n} = (\sqrt[n]{a^m}).$$

If n is even, then we require that $a \geq 0$.

It is not too difficult to show that the laws of exponents hold for rational exponents.
if $x = \frac{m_1}{n_1}$ and $y = \frac{m_2}{n_2}$ are rational numbers as above, we can show that

1. $a^x a^y = a^{x+y}$, (where $a \geq 0$ if n_1 or n_2 are even.)
2. $\frac{a^x}{a^y} = a^{x-y}$, $a \neq 0$, (where $a \geq 0$ if n_1 or n_2 are even.)
3. $(a^x)^y = a^{xy}$, (where $a \geq 0$ if n_1 or n_2 are even.)
4. $(ab)^x = a^x b^x$, (where $a, b \geq 0$ if n_1 is even.)

5. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$, $b \neq 0$. (where $a, b \geq 0$ if n_1 is even.)

In Calculus II, we will define a^x more generally when x is a real number and $a > 0$. The above laws of exponents will continue to hold for for these exponents. Note that the laws for integers and nth roots are special cases of the above laws of exponents.

Example Simplify the following:

$$\left(\frac{2x^{1/4}}{y^{1/5}}\right)^5 \left(\frac{y^2}{x}\right), \qquad \sqrt{x^3\sqrt{x}}$$