Lecture 4 : Rational Expressions

A quotient of two algebraic expressions is called a fractional expression. For example:

$$\frac{2x^2 - x + 1}{\sqrt{x} + 1} \qquad \frac{\sqrt{x} + 3}{x + 2} \qquad \frac{\sqrt{h+1} - 1}{h}.$$

A **rational expression** is a quotient of two polynomials. For example:

$$\frac{2x^2 - x + 1}{x + 1} \qquad \frac{x^3 + 3x^2 - x + 1}{x^2 + 1}$$

It is crucial to master algebraic manipulations of such expressions before studying calculus.

We start with the idea of the **domain** of such an expression which is related to the concept of the domain of a rational function which is treated later.

The Domain of an algebraic expression is the set of all real numbers that the variable is permitted to have. when finding domains, It is good to keep in mind that $\sqrt{x}$ is only defined for values of x which are ≥ 0 and $1/x$ not defined if $x = 0$.

Example Find the domain of the following algebraic expressions:

$$\frac{x}{x^2 - 6x + 9} \qquad \frac{\sqrt{x+1}}{x - 5}$$

Algebra of Rational Expressions

We can simplify rational expressions by **canceling factors** with this rule

$$\boxed{\frac{AC}{BC} = \frac{A}{B}}$$

Example Simplify the rational expression:

$$\frac{x^2 - 9}{x^2 - 4x + 3}$$

We can **multiply rational expressions** according to the rule

$$\boxed{\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}}$$

Example Multiply and simplify the following rational expressions

$$\frac{x^2 - 2x + 1}{x - 3} \cdot \frac{x^2 - 9}{x - 5x + 4}$$

To **divide rational expressions**, we use the following rule:

$$\boxed{\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}}$$

Example Divide and simplify the following rational expressions

$$\frac{x - 3}{x^2 - 4} \div \frac{x^2 - 9}{x^2 - 7x + 12}$$

To **add rational expressions** we use the rules

$$\boxed{\frac{A}{C} + \frac{B}{C} = \frac{A + B}{C}}$$

$$\boxed{\frac{A}{C} + \frac{B}{D} = \frac{AD + BC}{CD}}$$

In the latter case, we can simplify the calculation by using the Least Common Multiple of C and D.

Example Add the following rational expressions and simplify the resulting rational expression

$$\frac{x - 1}{x - 3} + \frac{1}{x - 3} \qquad \frac{x - 1}{x - 3} - \frac{1}{x^2 - 9} \qquad \frac{1}{x + h} - \frac{1}{x}$$

Simplifying a compound fraction Sometimes algebraic expressions involve rational expressions in both the numerator and denominator.

Example Simplify the following fractions:

$$\frac{\frac{1}{x+1} - 1}{1 + \frac{1}{x-2}} \qquad \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

Rationalizing the numerator or denominator This is common method used in calculation in calculus. The idea is to rationalize an expression using the fact that $(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B}) = A - B$.

Example Rationalize the denominator and numerator respectively in the following expressions:

$$\frac{1}{\sqrt{x} - 1} \qquad \frac{\sqrt{9+h} - 3}{h}$$