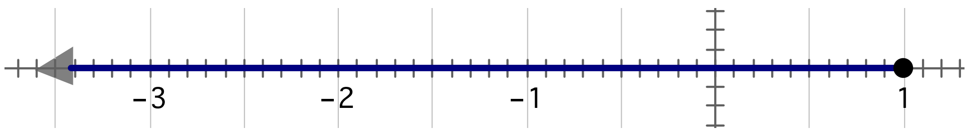
Lecture 7 : Inequalities

Sometimes a problem may require us to find all numbers which satisfy an inequality. An inequality is written like an equation, except the equals sign is replaced by one of the symbols, $<$, $\leq$, $>$, $\geq$.

Example $2x + 1 \leq 3$ is an inequality.

The solution of an inequality is the set of all numbers which satisfy the inequality. This set may have infinitely many numbers and may be represented by an interval or a number of intervals on the real line.

Example The solution to the inequality $2x + 1 \leq 3$ is the set of all $x \leq 1$.



The solution to the equation $2x + 1 = 3$ is the unique value $x = 1$.

To help solve inequalities, we use the following algebraic rules. They are much the same as the rules for manipulating equalities with one important difference in Rule number 4. Below, we assume that the symbols A, B and C stand for real numbers or algebraic expressions. The symbol $\iff$ means "if and only if".
as noted in Lecture 5, the term "if and only if" is a frequently used logical statement and it means that if the statement on the left hand side is true then the statement on the right hand side is guaranteed to be true also and vice-versa (if the statement on the right is true, then the statement on the left is guaranteed to be true.)
The rules are stated below for the inequality $\leq$. The rules hold if we replace $\leq$ by any of the inequalities $<$, $>$, or $\geq$.

Rule	**Description**
1. $A \leq B \iff A + C \leq B + C$	We can **add** the same quantity to both sides of the inequality to get an equivalent inequality
2. $A \leq B \iff A - C \leq B - C$	We can **subtract** the same quantity to both sides of the inequality to get an equivalent inequality
3. If $C > 0$ then $A \leq B \iff CA \leq CB$	We can **multiply** both sides of the inequality by the same positive number to get an equivalent inequality
4. If $C < 0$ then $A \leq B \iff CA \geq CB$	If we **multiply** both sides of the inequality by the same negative number, this reverses the direction of the inequality
5. If $A > 0$ and $B > 0$ and $A \leq B$, then $\dfrac{1}{A} \geq \dfrac{1}{B}$	If we **take reciprocals** of each side of an inequality involving positive quantities this reverses the direction of the inequality
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$	Inequalities can be added.

Linear Inequalities

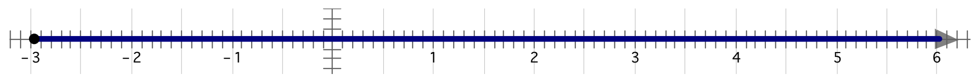
Example Solve the inequality $2x + 1 \leq 3$.

Apply Rule 2 by subtracting 1 from both sides $\quad 2x + 1 - 1 \leq 3 - 1 \quad \rightarrow \quad 2x \leq 2$
Apply Rule 3 by multiplying both sides by $\frac{1}{2} > 0 \quad \frac{1}{2}2x \leq \frac{1}{2}2 \quad \rightarrow \quad x \leq 1$
We can represent this set of solutions as an interval (see above).

Example Solve the inequality $2x + 1 \leq 5x + 10$.

Apply Rule 2 by subtracting 1 from both sides	$2x + 1 - 1 \leq 5x + 10 - 1$	$\rightarrow$	$2x \leq 5x + 9$
Apply Rule 2 by subtracting $5x$ from both sides	$2x - 5x \leq 5x + 9 - 5x$	$\rightarrow$	$-3x \leq 9$
Apply Rule 4 by multiplying both sides by $\frac{-1}{3} < 0$	$\frac{-1}{3}(-3)x \geq \frac{-1}{3}(9)$	$\rightarrow$	$x \geq -3$

We can represent this set of solutions as an interval:



Example Solve the inequality $5x - 1 \geq 10x - 46$.

Solving a pair of simultaneous inequalities

Sometimes we want to find values of x which satisfy two (or more) inequalities simultaneously. In this case, the solution set is all values of x which satisfy BOTH inequalities.

Example Solve the pair of simultaneous inequalities

$$-3 \leq 2x + 1 < 4$$

Example Solve the pair of simultaneous inequalities

$$2 - x < x, \quad x < 0$$

(here our inequalities are already written separately.)

Nonlinear Inequalities

To solve inequalities involving squares and other powers of the variable, we can sometimes use factorization and the following rule:

The sign of a product or quotient

- If a product or quotient has an even number of negative factors, then its value is positive.
- If a product or quotient has an odd number of negative factors, then its value is negative.

Example: A Quadratic Inequality Solve the inequality $x^2 + 3 \leq 4x$.
First we move all of the terms to one side:

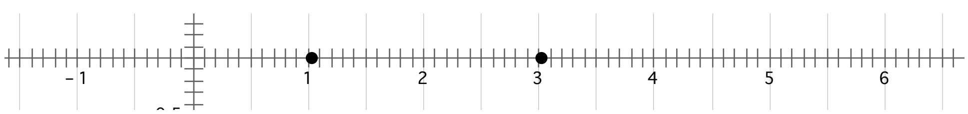
$$x^2 - 4x + 3 \leq 0$$

We then factor the non-zero side:

$$(x - 3)(x - 1) \leq 0$$

The expression on the left can only change sign if one of the factors changes sign. This happens (for a linear factor) when the factor is zero. So we find the points where the factors are zero. These points separate the number line into intervals. The sign of the expression on the left must remain the same on these intervals.

We have $x - 3 = 0$ when $x = 3$ and $x - 1 = 0$ when $x = 1$.

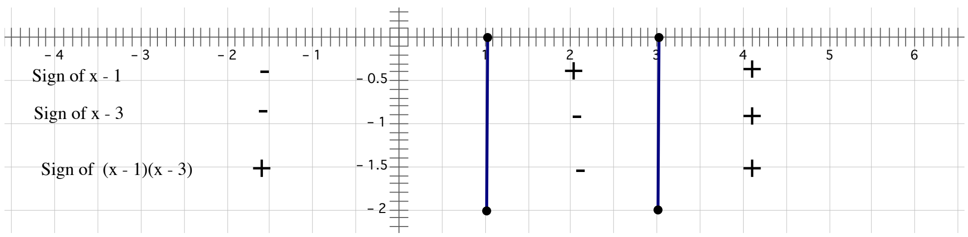


We now check the sign of the expression $(x-3)(x-1)$ on the intervals between the above points, $(-\infty, 1)$, $(1,3)$, $(3,\infty)$. We have two methods of doing this.

Method 1: Check the sign of each factor on each interval We have

Interval $\longrightarrow$	$(-\infty, 1)$	$(1,3)$	$(3,\infty)$
Sign of $x-1$	$-$	$+$	$+$
Sign of $x-3$	$-$	$-$	$+$
Sign of $(x-1)(x-3)$	$+$	$-$	$+$

We can represent this information on a picture as follows:



Our problem was to find where $(x-1)(x-3) \leq 0$. On the interval $(1,3)$, we have $(x-1)(x-3) < 0$ and $(x-1)(x-3) = 0$ at the endpoints, 1 and 3. Hence, we have $(x-1)(x-3) \leq 0$ on the closed interval $[1,3]$.

Method 2: Test the sign of the expressions at test points in each interval. Here we pick a point in each interval, plug that value in for x in the expression $(x-3)(x-1)$ and check the sign of the result. Since we know that this expression cannot change sign on the given intervals, this tells us the sign of the expression $(x-3)(x-1)$ on the entire interval.

Interval $\longrightarrow$	$(-\infty, 1)$	$(1,3)$	$(3,\infty)$
Test point	$x=-2$	$x=2$	$x=4$
Value of $(x-1)(x-3)$ at test point	$(-2-1)(-2-3) = 15$	$(2-1)(2-3) = -1$	$(4-1)(4-3) = 3$
Sign of $(x-1)(x-3)$ at test point	$+$	$-$	$+$
Sign of $(x-1)(x-3)$	$+$	$-$	$+$

Example Solve the inequality $x^2 - 5 \leq -4x$.

Quadratics which do not have linear factors

Recall that the quadratic ax^2+bx+c may factor into distinct linear factors, repeated linear factors (times a constant) or may not factor (already irreducible) depending on whether the discriminant $D = b^2 - 4ac$ is $> 0, = 0$ or < 0 respectively. Solving inequalities involving quadratics which do not have distinct linear factors (after you bring all terms to one side) is not difficult:

- If we have repeated roots, we should keep in mind that $(x-k)^2$ is always positive, since it is a square; hence the set of solutions to the inequality $(x-k)^2 \geq 0$ is the set of all real numbers and the set of solutions to the inequality $(x-k)^2 < 0$ is the empty set, since no value of x will give a negative value for $(x-k)^2$. We have $(x-k)^2 = 0$ when $x = k$.

- If $ax^2 + bx + c$ is irreducible with discriminant $D = b^2 - 4ac < 0$, then the graph of the function never crosses the x axis since there is no value of x for which $ax^2 + bx + c = 0$. This means that the graph $y = ax^2 + bx + c$ (a parabola) is either always above the x- axis or always below the x axis and the values of $ax^2 + bx + c$ are either always > 0 or always < 0. To determine which, you can check the value of the function when $x = 0$. (Here we are actually using the fact that the graph of the function has a property called continuity and a theorem called the Intermediate value Theorem to deduce the graph stays above or below the x axis. You will learn more about this property and theorem in Calculus 1).

Example Solve the inequality $4x - 4 \leq x^2$.

Example Solve the inequality $x^2 + x + 1 \leq 0$.

Inequalities involving rational Functions

The same process works for rational functions and more general non-liner products. Basically the steps in the process are:

- Move all terms to one side.
- Factor the non-zero side.
- Find the points where the factors are zero.
- Identify the connected intervals between these points.
- Make a table or diagram to identify where each factor (or the test values in each interval) is positive or negative.
- Identify the intervals which solve your inequality, including the endpoints if appropriate.

Example Solve the inequality $\dfrac{3+x}{1-x} \geq 1$.

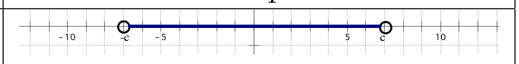
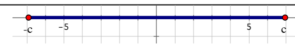
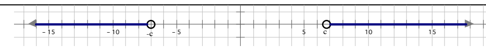
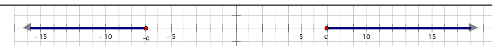
Example Solve the inequality $x - \dfrac{8}{x+1} < 1$.

Inequalities with absolute values

Recall that

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

This means that although $-4 < 1$, we have $|-4| = 4 > 1$. Hence we must consider the cases of negative values and positive values separately when dealing with absolute values. Below we show how to translate an inequality containing an absolute value to either one or two inequalities with no absolute value. We assume that c is a number with $c > 0$.

Inequality	Equivalent Form	Graph
$\|x\| < c$	$-c < x < c$	
$\|x\| \leq c$	$-c \leq x \leq c$	
$\|x\| > c$	$x < -c$ or $x > c$	
$\|x\| \geq c$	$x \leq -c$ or $x \geq c$	

These properties also hold with x replaced by A, where A is an algebraic expression in x.

Inequality	Equivalent To
$\|A\| < c$	$-c < A$ and $A < c$ (simultaneous)
$\|A\| \leq c$	$-c \leq A$ and $A \leq c$ (simultaneous)
$\|A\| > c$	$A < -c$ or $A > c$ (not simultaneous)
$\|A\| \geq c$	$A \leq -c$ or $A \geq c$ (not simultaneous)

Here is a typical example of an inequality you might have to solve in Calculus II when working with power series.

Example Solve the inequality $\dfrac{|x-3|}{2} < 4$.

Example Solve the inequality $|\frac{x-3}{x}| \geq 4$

If we have an inequality of the form $|A| \leq B$, where BOTH A and B are algebraic expressions in x, we must consider the cases where $A \geq 0$ and $A < 0$ separately. In each case we get a pair of simultaneous inequalities to solve, namely:
Case 1: Solve the simultaneous inequalities $A \leq B$ and $A \geq 0$
Case 2: Solve the simultaneous inequalities $-A \leq B$ and $A < 0$
The solution to the original inequality is the union of the solutions to case 1 and case 2.

The same method works for inequalities of the form $|A| \geq B$, $|A| < B$, $|A| > B$.

Example Solve the inequality $|x - 2| < 2x$.

Example Solve the inequality $|x+1| \geq \dfrac{x^2}{4}$.

Modeling with inequalities

Example The gas mileage g measured in (mi/gal) for a particular vehicle, driven at v mi/h, is given by the formula $g = 10 + 0.9v - 0.01v^2$, as long as v is between 10 mi/hr and 45 mi/hr. For what range of speeds is the vehicle's mileage 30 mi/gal or better?

Here is an example of modeling with inequalities that you may have to deal with when studying optimization in Calculus I.

Example A determined gardener has 120 feet of deer resistant fence. She wants to enclose a rectangular vegetable garden in her backyard and she wants the area enclosed to be 800 ft^2. What range of values is possible for the length of her garden?