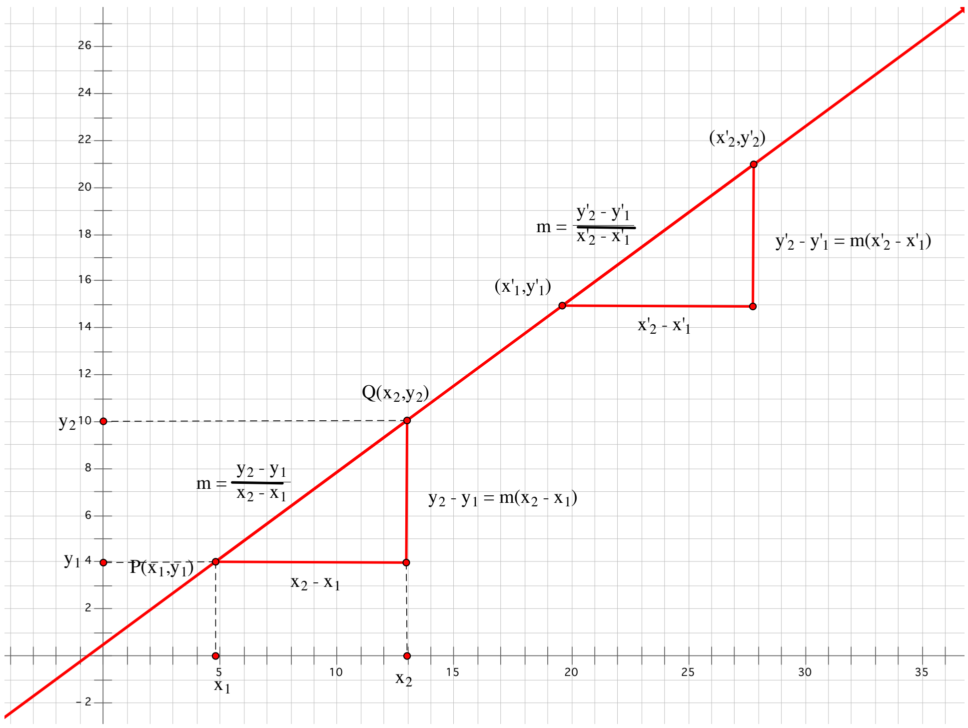
Lecture 9: Lines

If we have two distinct points in the Cartesian plane, there is a unique line which passes through the two points. We can construct it by joining the points with a straight edge and extending the line. Given any two points on a non-vertical line, the **slope**, m, of the line is the change in vertical distance between the two points divided by the change in horizontal distance between the two points.

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

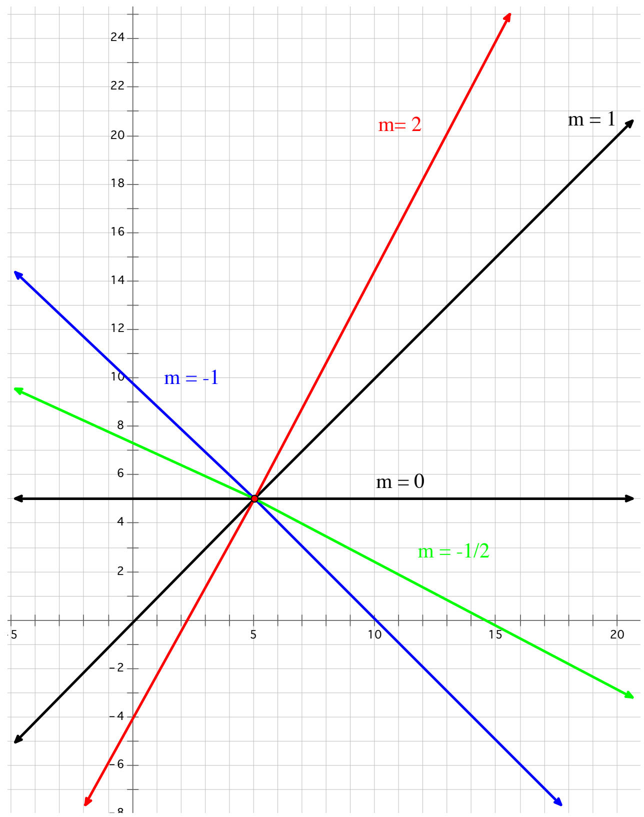
This slope remains the same no matter which two distinct points we choose on the line to calculate it (because the triangles below are similar).



Example Find the slope of the line that runs through the points $(1, 2)$ and $(4, 1)$.

Example Find the slope of the line that runs through the points $(0, 1)$ and $(-3, -5)$.

Likewise, if we are given a point P on the plane and a slope m, there is a unique line through the point P with that given slope. Below we show a number of lines through a point P with different slopes. Note that a horizontal line has a slope of 0, a line which is slanting upwards as we move from left to right has a **positive slope** and a line which is slanting downwards as we move from left to right has a **negative slope**.



Equation of the line through the point $P(x_1, y_1)$ with slope m

If (x, y) is any point on the line through $P(x_1, y_1)$ with slope m, then using the formula for the slope resulting from two points on the line and the given slope, we get an equation

$$\frac{y - y_1}{x - x_1} = m \quad \text{or} \quad \boxed{y - y_1 = m(x - x_1)}.$$

Example Find the equation of the line which passes through the point $(2, 3)$ and has slope 2.

Example Find the equation of the line which passes through the point $(-1, -2)$ and has slope -1.
Does this line cut through the origin?
Does this line pass through the point $(1, -2)$?

Example Find the equation of the line which passes through the points $(-1, 2)$ and $(1, 1)$.

Example Find the equation of the line which passes through the points $(0, 1)$ and $(3, -1)$. Does this line pass through the point $(-1, 2)$?

Slope-intercept form

If a line is not a vertical line, it must have a $y-$intercept. If the coordinates of that $y-$intercept are $(0, b)$ and the line has slope m, we have the equation of the line is (using the formula above)

$$y - b = mx \qquad \text{or} \qquad \boxed{y = mx + b}.$$

This is called the **slope-intercept form** of the equation. If the equation is in this form, we can identify the slope and the $y-$intercept directly from the equation. Note also that any equation of this form gives the equation of a line (the one which passes through the point $(0, b)$ with slope m).

Example Find the equation of a line with slope -2 and $y-$intercept 3.

General Equation of a line

In general, the equation of a line is given by

$$\boxed{Ax + By = C}$$

where A, B and C are constants.
(Every equation of this form where $B \neq 0$ describes a line since it can be transformed to one of the form $y = mx + b$, which we know (from above) describes a line. Conversely, every non-vertical line has an equation of the form $y = mx + b$ which fits the above description with $B = 1$, $A = -m$ and $C = b$. If $B = 0$, we can rearrange the equation to the form given below describing a vertical line.)

If the line is **vertical**, all of the x values on the line are the same (see example below) and its equation is of the form

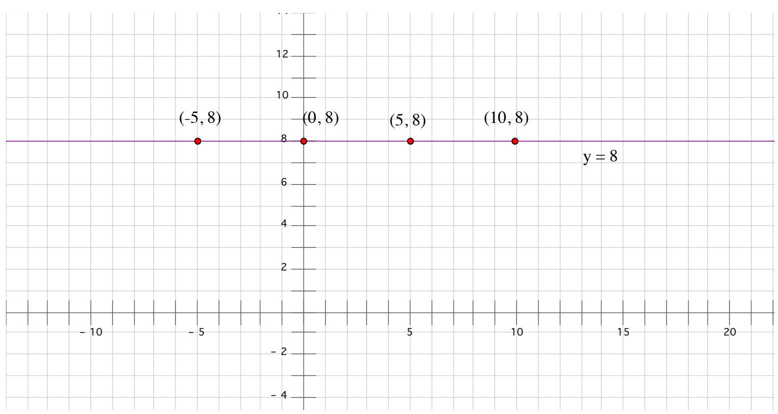
$$\boxed{x = D}$$

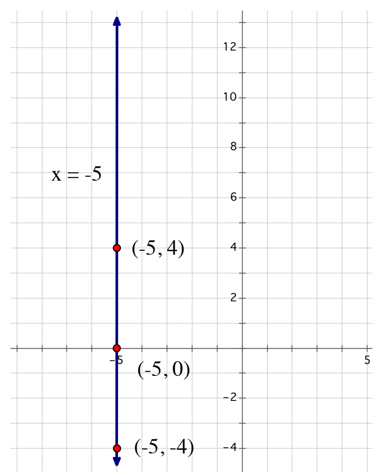
for some constant D. (This fits the above description with $B = 0$, $A = 1$ and $C = D$.)

If the line is **horizontal**, all of the y values on the line are the same (see example below) and its equation is of the form

$$\boxed{y = E}$$

for some constant E.





Sketching the graph of a line To sketch the graph of a line, it is enough to find two points on the line (the x and y-intercepts are usually the easiest when they exist) and draw the unique line joining those points.
Example Sketch the graphs of the following lines:

$$y = 5, \qquad x = -3, \qquad 8x + 2y = 10 \qquad 5x + y = 30$$

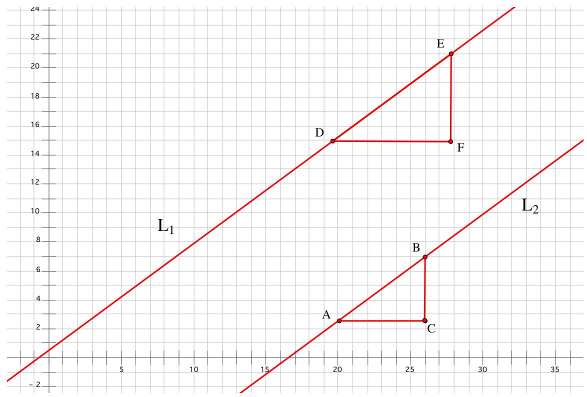
Parallel Lines

$\boxed{\text{Two lines are parallel if and only if they have the same slope.}}$

We can see this from the picture below. If both lines are parallel, the triangles ABC and DEF have the same angles and are similar. Therefore we have equal slopes, since

$$m_1 = \frac{d(E, F)}{d(D, F)} = \frac{d(B, C)}{d(A, C)} = m_2.$$

On the other hand, if the slopes are equal, the triangles will be similar (see the definition in Lecture 6) because $\frac{d(E,F)}{d(D,F)} = \frac{d(B,C)}{d(A,C)}$ implies that $\frac{d(E,F)}{d(B,C)} = \frac{d(D,F)}{d(A,C)}$ and $\angle$ DFE=$\angle$ ACB (which is a right angle). This implies that the angles $\angle$ EDF is equal to the angle $\angle$ BAC and therefore the lines are parallel.



Example Show that the lines $x + 2y = 1$ is parallel to the line $y = -\frac{1}{2}x$.

Example Give the equation of a line passing through the point $(0, 1)$ which is parallel to the line $x + 2y = 1$.

Perpendicular Lines

Two lines are perpendicular if they meet at right angles. We use the notation $L_1 \perp L_2$ to denote that the line L_1 is perpendicular to the line L_2. The slopes of perpendicular lines are related as follows:

Two lines with slopes m_1 and m_2 are perpendicular if and only if the product of their slopes is -1, that is

$$m_1 = -\frac{1}{m_2} \quad \text{or} \quad m_1 m_2 = -1.$$

Also a horizontal line is perpendicular to a vertical line.

Example Show that the lines $2y + x + 4 = 0$ is perpendicular to the line $y - 2x + 6 = 0$.

Example Find the equation of a line which is perpendicular to the line $x + 2y = 1$ which passes through the point $(0, 1)$.

Families of lines

Sometimes we consider families of curves in calculus, especially when we talk about antiderivatives and solutions to differential equations. Here we will look at families of lines. We first look at a finite set of lines.

Example Graph the set of lines $y = 2x + b$ where $b = -2, -1, 0, 1, 2$.

Example Find equations for a new set of lines $y = mx + a$ which intersect all of the above lines at right angles (called an orthogonal set of lines).

Example: An infinite family of lines What does the family of lines $y = 2x + b$ where b is any integer look like?

Example Orthogonal lines Find an infinite family of lines which are perpendicular to all of the above lines.

Slope as the Rate of Change

Here we look at a number of examples where a constant rate of change of some variable with respect to time allows us to model a situation with a linear function(line). In this case, the slope of the line is the constant rate of change.

Example Water is pumped out of a tank at the rate of 10 gallons per minute. The tank contains 500 gallons of water at the start of the process. Let y denote the amount of water in the tank at time t (minutes).

(a) Find the equation showing the (linear) relationship between y and time t?

(b) For what values of t is this relationship valid?

(c) Graph this relationship.

(d) Use the linear model to determine how much water will be in the tank after 11 minutes.

Example: Temperature Scales The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the equation

$$F = \frac{9}{5}C + 32.$$

(a) Find the temperature in Fahrenheit corresponding to the following temperature readings in Celsius:

$$-30^oC, \qquad -10^oC, \qquad 0^oC, \qquad 10^oC, \qquad 25^oC$$

(b) Find the temperature in Celsius corresponding to the following temperature readings given in Fahrenheit:

$$-30^oF, \qquad 0^oF, \qquad 25^oF, \qquad 90^oF$$

(c) Sketch the relationship between both scales on a Cartesian plane.

(d) Find the temperature at which both scales agree.

Distance, Speed and Time It an object travels at a constant speed, the speed is given by the formula

$$\text{speed} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

If the object is not traveling at a constant speed, this formula gives the average speed over the time period in question.

Example A car traveling at a constant speed leaves Mathville at 1:00 p.m. and reaches Calcville, 60 miles from Mathville, at 1:50 p.m.

(a) At what speed was the car traveling?

(b) Let D denote the distance traveled by the car from Mathville at time t (minutes after the car left Mathville). Write down the linear relationship between D and t.

(c) For which values of t is the above relationship valid (based on the information given). Draw a graph of this linear relationship in the Cartesian Plane.